Design of a Desirable Trajectory and Convergent Control for 3-D.O.F Manipulator with a Nonholonomic Constraint

T. Yoshikawa*, K. Kobayashi**, and T. Watanabe*

*Department of Mechanical Engineering
Kyoto University
Kyoto, 606-8501, Japan
{yoshi,watanabe}@mech.kyoto-u.ac.jp

**Department of Systems and Human Science
Osaka University
Osaka, 560-8531, Japan
kobayasi@sys.es.osaka-u.ac.jp

Abstract

This paper is concerned with control of a 3 link planar underactuated manipulator whose most distal joint is unactuated. This system is known as a second order nonholonomic system. In a previous paper, we proposed a control law that guarantees the convergence of its state to a given desirable trajectory and to any desired final point. We also gave a design method of the desirable trajectory, but this method has a limitation on the location of the initial state. In the present paper, we propose a design method of a desirable trajectory that starts from any given initial point, converges to any given desired final point, and on the way passes through any given desired passing point that can be specifyed rather freely. By this new design method, we can derive a desirable trajectory that satisfies given requirements much better than the previous method.

1 Introduction

Recently, there has been a growing interest in the control of nonholonomic systems. There are two important classes in nonholonomic systems. One is the class of first-order nonholonomic systems and the other is the class of second-order nonholonomic systems. The former systems have velocity-dependent constraints that are not integrable to obtain configuration-dependent constraints. Wheeled mobile robots, multifingered robot hands with rolling contact, and free-flying space robots are included in this class. The latter systems have acceleration-dependent constraints which are not integrable to obtain velocity/configuration-dependent constraints. Underactuated planar manipulators in which some joints are unactuated, submarine robots, and surface vessels are included in this class.

In the first- and second-order nonholonomic systems, there exist some systems which have the following two properties; (i) the linearization of the systems are not controllable, and (ii) there exists no time-invariant state feedback law to stabilize the systems [1] ∼[11]. First-order nonholonomic systems with these two properties have been studied by many researchers and various results about their controllability and stabilization have been obtained [1]∼[3]. A second-order nonholonomic system with the above two properties also have been studied , but the obtained results are still limited [4]∼[11].

One group of well-known second-order nonholonomic systems is a underactuated planar manipulator [5]∼ [11]. This group of the systems is more suitable for the mechanical analysis and verification than other systems such as submarine robots and surface vessels, since the equations of motion of the underactuated planar manipulators don't need obvious linearizing approximation. For the 2 link planar manipulator whose first joint (i.e., on the base side) is actuated and whose second joint (i.e., on the end-effector side) is unactuated, several closed-loop control methods have been developed [9]∼[11]. However, the controllability of the system has not been proved yet and the controller which guarantees the convergence to a desired final point has not been developed yet.

The controllability of the 3 link planar manipulator (including their systems) whose first and second joint (i.e., on the base side) is actuated and whose third joint (i.e., on the end-effector side) is unactuated has been proved by Arai et al. [6]. De Luca et al. [5] have formulated this system as a second-order chained form, have given a sufficient condition for the controllability, and have developed an open-loop controller which can achieve an any desired configuration. But, they haven't developed closed-loop controller. Arai et al. [7] have developed the design method of a trajectory from an any given initial state to an any desired final state and have given a closed-loop controller to converge its state to the trajectory. Arai et al.'s method of trajectory design is to determine two passing points as a function of the given initial and final states and to joint these four states by using circular and straight trajectories. However, they assumed that the initial state is at rest. For the case where the initial velocity is not zero, it needs to determine more passing points and the obtained trajectory is more complicated. Moreover, the method does not guarantee the convergence to the final state in the closed-loop control along the trajectory and has possibility that the control error will not be zero when control is over. Especially, it will be a big problem when a steady rotational velocity error in unactuated joint remains.

On the other hand, we have obtained a second-order chained form for the same system, and have proposed a closed-loop control method [8]. This method guarantees that the state converges exponentially to a desirable trajectory which converges exponentially to the origin. Therefore, the convergence of its state to a desirable trajectory and the exponential convergence to a desired final point are simultaneously obtained. One feature of this method is that it avoids the problem of generating steady states error from the desired final state by using a desirable trajectory whose final part consists of an exponential trajectory whose length is infinite. However, the desirable trajectory in [8] is limited, since the initial state of the trajectory cannot be given arbitrarily.

In this paper, we propose a new design method of a desirable trajectory that starts from any given initial state, passes through any given desired passing point, and converges exponentially to the origin, in order to control a 3 link planar manipulator whose first and second joint (i.e., on the base side) is actuated and whose third joint (i.e., on the end-effector side) is unactuated. We can use this trajectory as a desirable trajectory in the closed-loop controller proposed in [8]. The paper is organized as follows. In sections 2 and 3, the procedure to transform the equation of motion of the 3 link planar manipulator into a second-order chained form and the closed-loop controller for the chained form are briefly described [8]. Then, a new design method of a desirable trajectory for the controller is proposed in section 4. The validity of this method is illustrated by simulation results in section 5.

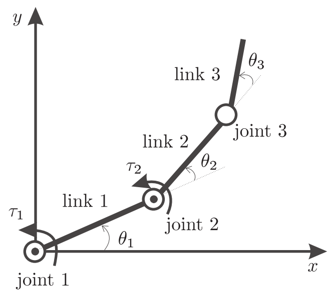


Figure 1: 3 link planar manipulator

2 Chained Form

We consider a manipulator shown in **Fig.1**. We assume that the manipulator moves in a plane and that gravity forces doesn't work. All joints are rotational ones, and we call the joints, joint 1, 2, 3, respectively, in the order of closeness to the base side. We also call the links, link 1, 2, 3, respectively, in a similar way. In addition, we assume that joints 1 and 2 are actuated and joint 3 is unactuated. Let $\theta_i (i = 1, 2, 3)$ be the each joint angle and $q = [\theta_1, \theta_2, \theta_3]^T$ be the generalized coordinates. We also let m_i the mass of link i, $\tilde{I}_i$ the inertia moment of link i, l_i the length of link i, l_{gi} the distance between joint i and the center of gravity of link i, andτ_i the torque of joint i. Then the equation of motion is given by

$$\begin{aligned} \tau_1 \;=\;& I_1\ddot{\theta}_1 + I_2(\ddot{\theta}_1 + \ddot{\theta}_2) + I_3(\ddot{\theta}_1 + \ddot{\theta}_2 + \ddot{\theta}_3) \\ & + (m_2 + m_3)l_1^2\ddot{\theta}_1 + m_3 l_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) \\ & + (m_2 l_{g2} + m_3 l_2)l_1\{C_2(2\ddot{\theta}_1 + \ddot{\theta}_2) \\ & - S_2(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2)\} \\ & + m_3 l_1 l_{g3}\{C_{23}(2\ddot{\theta}_1 + \ddot{\theta}_2 + \ddot{\theta}_3) \\ & - S_{23}(\dot{\theta}_2 + \dot{\theta}_3)(2\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)\} \\ & + m_3 l_2 l_{g3}\{C_3(2\ddot{\theta}_1 + 2\ddot{\theta}_2 + \ddot{\theta}_3) \\ & - S_3(2\dot{\theta}_1\dot{\theta}_3 + 2\dot{\theta}_2\dot{\theta}_3 + \dot{\theta}_3^2)\} \end{aligned} \qquad (1)$$

$$\begin{aligned} \tau_2 \;=\;& I_2(\ddot{\theta}_1 + \ddot{\theta}_2) + I_3(\ddot{\theta}_1 + \ddot{\theta}_2 + \ddot{\theta}_3) \\ & + m_3 l_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) \\ & + (m_2 l_{g2} + m_3 l_2)l_1(C_2\ddot{\theta}_1 + S_2\dot{\theta}_1^2) \\ & + m_3 l_1 l_{g3}(C_{23}\ddot{\theta}_1 + S_{23}\dot{\theta}_1^2) \\ & + m_3 l_2 l_{g3}\{C_3(2\ddot{\theta}_1 + 2\ddot{\theta}_2 + \ddot{\theta}_3) \\ & - S_3(2\dot{\theta}_1\dot{\theta}_3 + 2\dot{\theta}_2\dot{\theta}_3 + \dot{\theta}_3^2)\} \end{aligned} \qquad (2)$$

$$\begin{aligned} 0 \;=\;& I_3(\ddot{\theta}_1 + \ddot{\theta}_2 + \ddot{\theta}_3) + m_3 l_1 l_{g3}(C_{23}\ddot{\theta}_1 + S_3 2\dot{\theta}_1^2) \\ & + m_3 l_2 l_{g3}\{C_3(\ddot{\theta}_1 + \ddot{\theta}_2) + S_3(\dot{\theta}_1 + \dot{\theta}_2)^2\} \end{aligned} \qquad (3)$$

where, $I_i = \tilde{I}_i + m_i l_{gi}^2$, $C_i = \cos\theta_i$, $S_i = \sin\theta_i$, $C_{12} = \cos(\theta_1 + \theta_2)$, $C_{23} = \cos(\theta_2 + \theta_3)$, $S_{12} = \sin(\theta_1 + \theta_2)$, $S_{23} = \sin(\theta_2 + \theta_3)$, $C_{123} = \cos(\theta_1 + \theta_2 + \theta_3)$, and $S_{123} = \sin(\theta_1 + \theta_2 + \theta_3)$. Now we introduce the new coordinates c_x, c_y, and θ, determined by

$$\left\{\begin{array}{lcl} c_x &=& l_1C_1 + l_2C_{12} + \frac{I_3}{m_3l_{g3}}C_{123} \\ c_y &=& l_1S_1 + l_2S_{12} + \frac{I_3}{m_3l_{g3}}S_{123} \\ \theta &=& \theta_1 + \theta_2 + \theta_3 \end{array}\right. \tag{4}$$

Then, from (1), (2), and (3), we get

$$\left\{\begin{array}{l} \ddot{c}_x = \cos\theta v_1 \\ \ddot{c}_y = \sin\theta v_1 \\ \ddot{\theta} = v_2 \end{array}\right. \tag{5}$$

where v_1, v_2 are the new inputs satisfying

$$\begin{bmatrix} -\frac{C_{12}}{S_2l_1} & \frac{1}{S_2}(\frac{C_{12}}{l_1} - \frac{C_1}{l_2}) \\ -\frac{S_{12}}{S_2l_1} & \frac{1}{S_2}(\frac{S_{12}}{l_1} - \frac{S_1}{l_2}) \end{bmatrix} \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} - \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$
$$= \boldsymbol{W} \begin{bmatrix} C_{123} & -S_{123} \\ S_{123} & C_{123} \end{bmatrix} \begin{bmatrix} v_1 + \frac{I_3}{m_3l_{g3}}(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3)^2 \\ -\frac{I_3}{m_3l_{g3}}v_2 \end{bmatrix} \tag{6}$$

where, α_1, α_2, $\boldsymbol{W}$ are given by

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} -m_3l_{g3}\cos\theta\dot{\theta}^2 \\ -m_3l_{g3}\sin\theta\dot{\theta}^2 \end{bmatrix}$$
$$+ \begin{bmatrix} \frac{C_{12}}{S_2}\{\frac{I_1}{l_1} + m_2l_2(1 - \frac{l_{g2}}{l_2})\} & -\frac{C_1}{S_2}(\frac{I_2}{l_2} - m_2l_{g2}) \\ \frac{S_{12}}{S_2}\{\frac{I_1}{l_1} + m_2l_2(1 - \frac{l_{g2}}{l_2})\} & -\frac{S_1}{S_2}(\frac{I_2}{l_2} - m_2l_{g2}) \end{bmatrix}$$
$$\times \begin{bmatrix} \frac{1}{S_2l_1}\{C_2l_1\dot{\theta}_1^2 + l_2(\dot{\theta}_1 + \dot{\theta}_2)^2\} \\ -\frac{1}{S_2l_2}\{l_1\dot{\theta}_1^2 + C_2l_2(\dot{\theta}_1 + \dot{\theta}_2)^2\} \end{bmatrix}$$

$$\boldsymbol{W} = \begin{bmatrix} w_1 & w_2 \\ w_2 & w_3 \end{bmatrix}$$
$$+ \begin{bmatrix} \frac{C_{12}}{S_2}\{\frac{I_1}{l_1} + m_2l_2(1 - \frac{l_{g2}}{l_2})\} & -\frac{C_1}{S_2}(\frac{I_2}{l_2} - m_2l_{g2}) \\ \frac{S_{12}}{S_2}\{\frac{I_1}{l_1} + m_2l_2(1 - \frac{l_{g2}}{l_2})\} & -\frac{S_1}{S_2}(\frac{I_2}{l_2} - m_2l_{g2}) \end{bmatrix}$$
$$\times \begin{bmatrix} \frac{C_{12}}{S_2l_1} & \frac{S_{12}}{S_2l_1} \\ -\frac{C_1}{S_2l_2} & -\frac{S_1}{S_2l_2} \end{bmatrix}$$

$$w_1 = m_2\frac{l_{g2}}{l_2} + m_3 - \frac{m_3^2l_{g3}^2}{I_3}\sin^2\theta$$
$$w_2 = \frac{m_3^2l_{g3}^2}{I_3}\sin\theta\cos\theta$$
$$w_3 = m_2\frac{l_{g2}}{l_2} + m_3 - \frac{m_3^2l_{g3}^2}{I_3}\cos^2\theta$$

Note that the new coordinates c_x and c_y express the center of collision [6], and θ expresses the angle between link 3 and x axis. Note also that v_1 and v_2 ,respectivity, are the force which acts on the center of collision in the direction whose angle to x axis is θ and the torque which acts on the center of collision.

Subsequently, using another coordinate and input transformations given by

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} c_x - \frac{I_3}{m_3l_{g3}} \\ \tan\theta \\ c_y \end{bmatrix} \tag{7}$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \sec\theta u_1 \\ \cos^2\theta u_2 - 2\tan\theta\dot{\theta}^2 \end{bmatrix} \tag{8}$$

we get

$$\left\{\begin{array}{l} \ddot{\xi}_1 = u_1 \\ \ddot{\xi}_2 = u_2 \\ \ddot{\xi}_3 = \xi_2u_1 \end{array}\right. \tag{9}$$

(9) is called the second-order chained form. Note that we can also get (9) when we use $\xi = [c_x,\ \tan\theta,\ c_y]^T$ in place of ξ given by (7). Then, $[\xi_1\ ,\ \xi_3]$ corresponds to the center of collision, and ξ_2 corresponds to the tangent of the angle between link 3 and x axis. But, in order that the proximal end position of link 3 and the angle between link 3 and x axis, respectivity, correspond to the origin and 0 when $\xi = 0$, we use ξ_1 in (7). Note that I_3/m_3l_{g3} is the distance between the proximal end and the center of collision of link 3. (this is to show the convergent (desired) configuration in the simulation in section 5 easier.) Note also that we can let $\xi = 0$ correspond to any static state by using a similar coordinate transformation [8]. So, the problem for the control of the system which guarantees the convergence of its state to any static state can be replaced by the problem of the convergence of ξ to 0.

3 Controller

We have already the controller . In this section, we summarize the controller proposed in [8] for the system given by (9).

We consider the following control inputs.

$$\left\{\begin{array}{lcl} u_1 &=& \ddot{r}_1(t) - k_1(t)(\dot{\xi}_1 - \dot{r}_1(t)) \\ && -\ k_2(t)(\xi_1 - r_1(t)) \\ u_2 &=& \ddot{r}_2(t) - k_3(t)(\dot{\xi}_2 - \dot{r}_2(t)) \\ && -\ k_4(t)(\xi_2 - r_2(t)) \\ && -\ \frac{k_5(t)(\dot{\xi}_3 - \dot{r}_3(t))}{\ddot{r}_1(t)} \\ && -\ \frac{k_6(t)(\xi_3 - r_3(t))}{\ddot{r}_1(t)} \end{array}\right. \tag{10}$$

Here, $k_i(t)$ is given by

$$k_i(t) = \kappa_{ij} \quad (t_j \leq t < t_{j+1}) \tag{11}$$

where κ_{ij} is determined to make the following matrices asymptotically stable.

$$\Lambda_{1j} = \begin{bmatrix} \lambda_j - \kappa_{1j} & -\kappa_{2j} \\ 1 & \lambda_j \end{bmatrix} \tag{12}$$

$$\tilde{\Lambda}_j = \begin{bmatrix} -\kappa_{3j} & -\kappa_{4j} & -\kappa_{5j} & -\kappa_{6j} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & \lambda_j & 0 \\ 0 & 0 & 1 & \lambda_j \end{bmatrix} \quad (13)$$

In addition, $r_i(t)(i = 1, 2, 3)$ is a desirable trajectory of $\xi_i(t)$ to specify the transient response, satisfying the following conditions.

(i) $\ddot{r}_1(t)$ and $\ddot{r}_2(t)$ are bounded.

(ii) For $t_j(j = 1, 2, \ldots, m)$ satisfying $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = \infty$,

$$\ddot{r}_1(t) = a_j e^{-\lambda_j(t-t_j)}(t_j \leq t < t_{j+1})$$

where $\lambda_j \geq 0(j = 0, 1, \ldots, m-1), \lambda_m > 0$.

(iii) $\lim_{t\to\infty} r_1(t) = 0$.

(iv) there exist a positive constant $\epsilon_i(i = 2, 3)$ satisfying $\lim_{t\to\infty} r_i(t)e^{(\epsilon_i+(i-2)\lambda_m)t} = 0$.

(v) $\ddot{r}_3(t) = r_2(t)\ddot{r}_1(t)$.

Note that $r_i(t)$ is a trajectory which converges exponentially to the origin due to conditions (ii), (iii), and (iv).

Then, we get the following theorem.

$\ll$ **Theorem1** $\gg$

Suppose a desirable trajectory $r(t) = [r_1(t),\ r_2(t),\ r_3(t)]^T$ for the system given by (9) satisfying the conditions (i)∼(v) is given. Let the state of the system be $x = [x_1^T, x_2^T, x_3^T]^T$ where $x_i = [\dot{\xi}_i(t), \xi_i(t)]^T(i = 1, 2, 3)$, and the error between the state and the desirable trajectory be $e = [e_1^T, e_2^T, e_3^T]^T$ where $e_i = [\dot{\xi}_i(t) - \dot{r}_i(t), \xi_i(t) - r_i(t)]^T(i = 1, 2, 3)$. Then, applying the controller given by (10) to the system, there exist a monotonous increasing and differentiable function ϕ satisfying $\phi(0) = 0$ and a positive constant α which satisfy

$$\| e(t) \| \leq \phi(\| e(0) \|)e^{-\alpha t} \quad (t \geq 0) \quad (14)$$

Theorem 1 guarantees that the state converges to the desirable trajectory and finally to the origin even when there exist some error between the initial state of the real system and the desirable trajectory. However, in [8], we only gave the trajectory of $r_1(t)$ explicitly, and let $r_2(t)$, $r_3(t)$ be $r_2(t) = r_3(t) = 0$. So, in the following section, we address the problem of designing a desirable trajectory with non-zero $r_2(t)$ and $r_3(t)$.

4 Desirable Trajectory

In this section, we consider the following problem.

[**Problem 1**] Suppose that an initial state $[\dot{\xi}_{di}(0), \xi_{di}(0)]^T(i = 1, 2, 3)$, a desired passing time t_d, and a desired passing state $[\dot{\xi}_{di}(t_d), \xi_{di}(t_d)]^T(i = 1, 2, 3)(\dot{\xi}_{d1}(t_d) = 0)$ are given for the system given by (9). Based on these data, design a desirable trajectory $r_i(t)(i = 1, 2, 3)$ satisfying conditions (i)∼(v).

Because of $\dot{r}_1(t_d) = 0$, $\xi_1(t_d)$ mostly becomes a switching point where the direction of ξ_1 changes. In [8], the desired passing time t_d and the desired passing point $\xi_{d1}(t_d)$ were given. But, since $\xi_{d1}(t_d)$ didn't correspond to the actual switching point because of $\dot{\xi}_{d1}(t_d) \neq 0$, it is to hard to understand the desirable trajectory given by [8] intuitionally. In addition, compared with a desirable trajectory given by Arai et al. [6] [7], this desirable trajectory seems better in the sense that the number of desired passing points is only one whether the initial state is at rest or not, and that we can select any point as the desired passing point.

In the following, we actually design the trajectory.

First, we design a trajectory $r_1(t)$. In order to satisfy conditions (i)∼(iii), suppose

$$\ddot{r}_1(t) = \begin{cases} a_0 & (0 \leq t < t_1) \\ a_1 & (t_1 \leq t < t_2) \\ a_2 & (t_2 \leq t < t_3) \\ a_3 \exp(-\lambda(t - t_3)) & (t_3 < t) \end{cases} \quad (15)$$

where

$$\begin{cases} t_1 &= \frac{t_d}{2} \\ t_2 &= t_d \\ t_3 &= t_d + \frac{2(\xi_{d1}(t_d)-h)}{\lambda h} \end{cases} \quad (16)$$

and

$$\begin{cases} a_0 &= \frac{4\xi_{d1}(t_d)-3t_d\dot{\xi}_{d1}(0)-4\xi_{d1}(0)}{t_d^2} \\ a_1 &= -\frac{4\xi_{d1}(t_d)-t_d\dot{\xi}_{d1}(0)-4\xi_{d1}(0)}{t_d^2} \\ a_2 &= -\frac{\lambda^2h^2}{2(\xi_{d1}(t_d)-h)} \\ a_3 &= \lambda^2 h \end{cases} \quad (17)$$

which are determined under the boundary conditions at t_1, t_2, and t_3. Note that $h = k\xi_{d1}(t_d)(0 < k < 1)$ is a design parameter satisfying $r_1(t_3) = h$.

Integrating (15), $r_1(t)$ is given by

$$r_1(t) = \begin{cases} \frac{a_0}{2}t^2 + \dot{\xi}_{d1}(0)t + \xi_{d1}(0) & (0 \leq t < t_1) \\ \frac{a_1}{2}(t-t_2)^2 + \xi_{d1}(t_d) & (t_1 \leq t < t_2) \\ \frac{a_2}{2}(t-t_2)^2 + \xi_{d1}(t_d) & (t_2 \leq t < t_3) \\ h\exp(-\lambda(t - t_3)) & (t_3 < t) \end{cases} \quad (18)$$

Since we consider the case $\ddot{r_1}(t) \neq 0$ which is a singular point of the controller given by (10), the desirable trajectory of $\xi_1(t)$, $r_1(t)$, is required that $\ddot{r_1}(t)$

has a large value even when $\| \xi_1(t) \|$ is smaller than $\| \xi_1(0) \|$. Determining $r_1(t)$ by using (18), we can get a practical trajectory which first goes to the position $\xi_1(t_d) = \xi_{d1}(t_d)$ and then converges to the origin even when $\xi_1(0) = 0$.

Next, we design $r_2(t)$ and $r_3(t)$. From (15), we get $\ddot{r}_1(t) \neq 0$. Hence, if we assume that ξ_1 and ξ_3 are (virtual) outputs, all states and inputs $[\xi\ ,\ \dot{\xi}\ ,\ u]$ become functions of $\xi_1^{(i)}$ and $\xi_3^{(i)}$ (i=0,1,2,...). This property is called flatness [12]. Concretely, from (9) and condition (ii), we can get

$$r_2(t) = \frac{\ddot{r}_3(t)}{\ddot{r}_1(t)} \quad (19)$$

$$\dot{r}_2(t) = \frac{(r_3^{(3)}(t) + \lambda_j \ddot{r}_3(t))}{\ddot{r}_1(t)} \quad (20)$$

for $t_j \leq t < t_{j+1}$. Because of the flatness property, the problem of designing desirable trajectories $r_2(t)$ and $r_3(t)$ satisfying conditions (iv) and (v) can be replaced by the problem of designing $r_3(t)$ which is differentiable 3 times for any interval $(t_j \leq t < t_{j+1})$. In the following part, we design $r_3(t)$ first for interval $(0 \leq t < t_d)$ and then for interval $(t_d \leq t)$.

First, $r_3(t)$ for interval $(0 \leq t < t_d)$ is given by

$$r_3 = \begin{cases} \sum_{k=0}^{5} A_k (t-t_1)^k & (t < t_1) \\ \sum_{k=0}^{5} B_k (t-t_1)^k & (t_1 \leq t < t_2 = t_d) \end{cases} \quad (21)$$

Here, we use a fifth-order time polynomial function, in order to satisfy that r_2, r_3, $\dot{r}_2$, and $\dot{r}_3$ are continuous at t_1, and to satisfy the boundary conditions at the initial and the desired passing states. A_k and B_k are given by

$$\begin{cases} A_0 = (4\gamma(a_0+a_1) + 6\delta(a_0-a_1))/\Delta \\ A_1 = (-10\gamma(a_0-a_1) - 20\delta(a_0+a_1))/\Delta \\ A_2 = 10\xi_{d3}(0) + 6\dot{\xi}_{d3}(0)t_1 + 1.5a_0\xi_{d2}(0)t_1^2 \\ \qquad + \frac{1}{6}a_0\dot{\xi}_{d2}(0)t_1^3 + 4A_1t_1 - 10A_0 \\ A_3 = 20\xi_{d3}(0) + 14\dot{\xi}_{d3}(0)t_1 + 4a_0\xi_{d2}(0)t_1^2 \\ \qquad + \frac{1}{2}a_0\dot{\xi}_{d2}(0)t_1^3 + 6A_1t_1 - 20A_0 \\ A_4 = 15\xi_{d3}(0) + 11\dot{\xi}_{d3}(0)t_1 + 3.5a_0\xi_{d2}(0)t_1^2 \\ \qquad + \frac{1}{2}a_0\dot{\xi}_{d2}(0)t_1^3 + 4A_1t_1 - 15A_0 \\ A_5 = 4\xi_{d3}(0) + 3\dot{\xi}_{d3}(0)t_1 + a_0\xi_{d2}(0)t_1^2 \\ \qquad + \frac{1}{6}a_0\dot{\xi}_{d2}(0)t_1^3 + A_1t_1 - A_0 \end{cases}$$

$$\begin{cases} B_0 = A_0 \\ B_1 = A_1 \\ B_2 = 10\xi_{d3}(t_d) - 6\dot{\xi}_{d3}(t_d)t_1 + 1.5a_0\xi_{d2}(t_d)t_1^2 \\ \qquad - \frac{1}{6}a_0\dot{\xi}_{d2}(t_d)t_1^3 - 4A_1t_1 - 10A_0 \\ B_3 = -20\xi_{d3}(t_d) + 14\dot{\xi}_{d3}(t_d)t_1 - 4a_0\xi_{d2}(t_d)t_1^2 \\ \qquad + \frac{1}{2}a_0\dot{\xi}_{d2}(t_d)t_1^3 + 6A_1t_1 + 20A_0 \\ B_4 = 15\xi_{d3}(t_d) - 11\dot{\xi}_{d3}(t_d)t_1 + 3.5a_0\xi_{d2}(t_d)t_1^2 \\ \qquad - \frac{1}{2}a_0\dot{\xi}_{d2}(t_d)t_1^3 - 4A_1t_1 - 15A_0 \\ B_5 = -4\xi_{d3}(t_d) + 3\dot{\xi}_{d3}(t_d)t_1 - a_0\xi_{d2}(t_d)t_1^2 \\ \qquad + \frac{1}{6}a_0\dot{\xi}_{d2}(t_d)t_1^3 + A_1t_1 + A_0 \end{cases}$$

where

$$\begin{cases} \gamma = a_1(20\xi_{d3}(0) + 14\dot{\xi}_{d3}(0)t_1 \\ \qquad + 4a_0\xi_{d2}(0)t_1^2 + \frac{1}{2}a_0\dot{\xi}_{d2}(0)t_1^3) \\ \qquad + a_0(20\xi_{d3}(t_d) - 14\dot{\xi}_{d3}(t_d)t_1 \\ \qquad + 4a_1\xi_{d2}(t_d)t_1^2 - \frac{1}{2}a_1\dot{\xi}_{d2}(t_d)t_1^3) \\ \delta = a_0(10\xi_{d3}(0) + 6\dot{\xi}_{d3}(0)t_1 \\ \qquad + 1.5a_0\xi_{d2}(0)t_1^2 + \frac{1}{6}a_0\dot{\xi}_{d2}(0)t_1^3) \\ \qquad - a_1(10\xi_{d3}(t_d) + 6\dot{\xi}_{d3}(t_d)t_1 \\ \qquad - 1.5a_1\xi_{d2}(t_d)t_1^2 + \frac{1}{6}a_0\dot{\xi}_{d2}(0)t_1^3) \\ \Delta = 20(a_0^2 + a_1^2) + 280a_0a_1 \end{cases}$$

Next, we design $r_3(t)$ for interval $(t_d \leq t)$. we use an exponential function as a trajectory of $r_3(t)$ for this interval. Using α $(\alpha > 2\lambda)$, $r_3(t)$ for interval $(t_d = t_2 \leq t < t_3)$ is given by

$$\begin{aligned} r_3(t) = {} & e^{-\alpha(t-t_2)} \\ & \times \{\xi_{d3}(t_d)(1+\alpha(t-t_2) \\ & + \frac{1}{2!}\alpha^2(t-t_2)^2 + \frac{1}{3!}\alpha^3(t-t_2)^3) \\ & + \dot{\xi}_{d3}(t_d)(1+\alpha(t-t_2) \\ & + \frac{1}{2!}\alpha^2(t-t_2)^2)(t-t_2) \\ & + \ddot{r}_3(t_2)(1+\alpha(t-t_2))\frac{1}{2!}(t-t_2)^2 \\ & + r_3^{(3)}(t_2)\frac{1}{3!}(t-t_2)^3\} \end{aligned} \quad (22)$$

and for interval $(t_3 \leq t)$ is given by

$$\begin{aligned} r_3(t) = {} & e^{-\alpha(t-t_3)} \\ & \times \{r_3(t_3)(1+\alpha(t-t_3) \\ & + \frac{1}{2!}\alpha^2(t-t_3)^2 + \frac{1}{3!}\alpha^3(t-t_3)^3) \\ & + \dot{r}_3(t_3)(1+\alpha(t-t_3) \\ & + \frac{1}{2!}\alpha^2(t-t_3)^2)(t-t_3) \\ & + \ddot{r}_3(t_3)(1+\alpha(t-t_3))\frac{1}{2!}(t-t_3)^2\} \\ & + r_3^{(3)}(t_3)\frac{1}{3!}(t-t_3)^3\} \end{aligned} \quad (23)$$

Here, $\ddot{r}_3(t_2)$ and $r_3^{(3)}(t_2)$ are given by (15), (19), (20), $\xi_{d2}(t_d)$, and $\dot{\xi}_{d2}(t_d)$. $r_3(t_3)$, $\dot{r}_3(t_3)$, $\ddot{r}_3(t_3)$, and $r_3^{(3)}(t_3)$ are given by (15), (19), (20), and (22) under the continuity of r_2, r_3, $\dot{r}_2$, and $\dot{r}_3$ at t_3.

Summarizing the above, the solution of Problem 1 is as follows.

1.we obtain $r_1(t)$ from (17) and (18).

2.we obtain $r_3(t)$ for interval $(0 \leq t < t_d)$ from (21), and $r_3(t)$ for interval $(t_d \leq t)$ from (22) and (23).

3.we obtain $r_2(t)$ from (19) and (20).

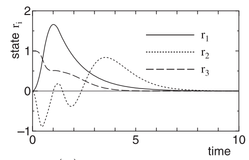
$r_i(t)$ given by the above design satisfies $r_i(0) = \xi_{di}(0)$, $r_i(t_d) = \xi_{di}(t_d)$, and conditions (i)$\sim$(v).

5 Simulation

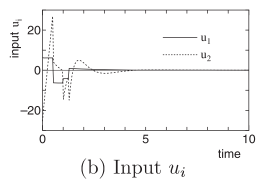
We show simulation results in this section to verify the validity of our approach. We design a desirable trajectory when the initial configurations are given by $[\theta_1(0),\theta_2(0),\theta_3(0)]^T = [150,\ -120,\ -30]^T$(degrees) and$[\dot{\theta}_1(0),\dot{\theta}_2(0),\dot{\theta}_3(0)]^T = [-0.1,\ 0,\ 0.1]^T$, and when $l_1 = l_2 = 1$, $l_3 = 0.5$, $l_{g1} = l_{g2} = 0.5$, $l_{g3} = 0.25$, $m_i = 1$, $I_1 = I_2 = 1/3$, and $I_3 = 0.25/3$.

The initial values are $[\xi_{d1}(0),\xi_{d2}(0),\xi_{d3}(0)]^T = [0{,}0{,}1]^T$ and $[\dot{\xi}_{d1}(0),\dot{\xi}_{d2}(0),\dot{\xi}_{d3}(0)]^T = [0.1{,}0{,}0]^T$. Let the values at the switching point be $[\xi_{d1}(t_d),\xi_{d2}(t_d),\xi_{d3}(t_d)]^T = [2{,}0{,}0.5]^T$ and $[\dot{\xi}_{d1}(t_d),\dot{\xi}_{d2}(t_d),\dot{\xi}_{d3}(t_d)]^T = [0{,}2{,}0]^T$, and the time at the switching point be $t_d = 1$. We also set $\lambda = 0.8$, $\alpha = 2.5$, and $h = 1.5$. The obtained trajectory is shown in **Fig.2**. Fig.2(a) shows a trajectory of the state $r_i (i = 1, 2, 3)$, Fig.2(b) shows the corresponding input $u_i (i = 1, 2)$, Fig.2(c) shows the behavior of link 3 where white and black circles, respectivity, are the proximal end and the tip positions of link 3 at each time. From these figures, we can see the convergence of the state of the system to the desired position and configuration (i.e., the origin).

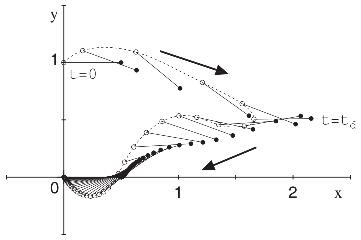
Next, in order to see the stability of the system, we did the simulation when the initial values are $[\xi_1(0),\xi_2(0),\xi_3(0)]^T = [0.1{,}0{,}1.1]^T$ and $[\dot{\xi}_1(0),\dot{\xi}_2(0),\dot{\xi}_3(0)]^T = [0{,}0{,}0]^T$, namely there exist an error between the real initial states of the real system and the desirable trajectory. The feedback gains are given by $k_1(t) = 4$, $k_2(t) = 5$, $[k_3(t),k_4(t),k_5(t),k_6(t)]$=[8,29,52,40] for $t < t_3$, and $[k_3(t),k_4(t),k_5(t),k_6(t)]$ =[9,38,88,74] for $t > t_3$. These gains have been determined from pole assignment of $\tilde{\Lambda}_{1j}$ and $\tilde{\Lambda}_j$. The result is shown in **Fig.3**. Fig.3(a) shows the response of $\xi_i (i = 1, 2, 3)$, and Fig.3(b) shows the behavior of link 3. From these figures, we can show the convergence of the state of the system to



(a) Trajectory r_i

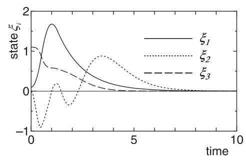


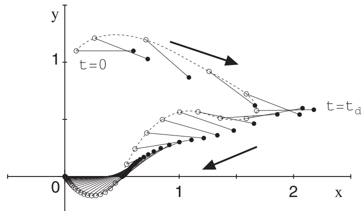
(b) Input u_i



(c) The behavior of link 3

Figure 2: Desirable trajectory





(a) Response of ξ_i (b) The behavior of link 3

Figure 3: Simulation results with initial error

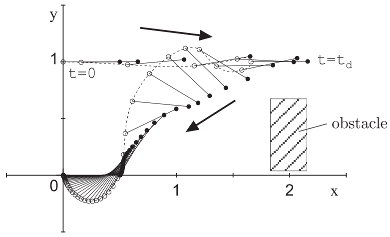


Figure 4: Desirable trajectory with a different switching point

the desirable trajectory and finally to the origin even when there exists an initial error.

We have also obtained another desirable trajectory shown in **Fig.4**, just changing the state at the switching point to $[\xi_{d1}(t_d),\xi_{d2}(t_d),\xi_{d3}(t_d)]^T = [2{,}0{,}1]^T$ and $[\dot{\xi}_{d1}(t_d),\dot{\xi}_{d2}(t_d),\dot{\xi}_{d3}(t_d)]^T = [0{,}2{,}0]^T$. Fig.4 shows that we can derive a desirable trajectory that satisfies some given requirements such as avoiding obstacles(for example, we can avoid the hatched obstacle shown in Fig.4 by the above change of the switching point.).

6 Conclusions

In this paper, we have propsed a design method of a desirable trajectory that starts from any given initial point, passes through any given desired passing point, and converges to any given desired final point, in order to control a 3 link planar manipulator with a nonholonomic constraint. We can use this trajectory as a desirable trajectory in the controller given by [8]. We have also presented simulation results in order to show the validity of this method. By this new design method, we can derive a desirable trajectory that satisfies given requirements much better than the previous method proposed in [8].

References

[1] Y. Nakamura and R. Mukherjee : " Nonholonomic Path Planning of Space Robots via a BiDirectional Approach", IEEE Transaction on Robotics and Automation, Vol. 7, No. 4, pp. 500–514, 1991

[2] M. Sanpei, T. Tamura, T. Kobayashi and N. Shibui : "Arbitrary Path Tracking Control of Articulated Vehicles Using Nonlinear Control Theory", IEEE Transaction on Control Systems Technology, Vol. 3, No. 1, pp. 125–131, 1995

[3] J. Imura, K. kobayashi and T. Yoshikawa : "Exponential Stabilization Problem of Nonholonomic Chained Systems with Specified Transient Response", Pro. of the 35th IEEE Conference on Decision and Control, pp. 4733–4738, 1996

[4] G. Oriolo and Y. Nakamura : "Free-Joint Manipulators: Motion Control under Second-Order Nonholonomic Constrains," IEEE/RSJ International Workshop on Intelligent Robots and Systems IROS'91, IEEE Cat. No. 91TH0375-6, pp. 1248–1253, 1991

[5] A. De Luca, R. Mattone and G. Oriolo : "Dynamic Mobility of Redundant Robots using End-Effector Commands", Proceeding of the 1996 IEEE International Conference on Robotics and Automation, Minneapolis, pp. 1760–1767, 1996

[6] H. Arai : "Controllability of a 3-DOF Manipulator with a Passive Joint under a Nonholonomic Constraint", Proceeding of the 1996 IEEE International Conference on Robotics and Automation, Minneapolis, pp. 3707–3713, 1996

[7] H. Arai, K. Tanie and N. Shiroma: "Feedback Control of a 3-DOF Planar Underactuated Manipulator ", Proceeding of the 1997 IEEE International Conference on Robotics and Automation, Albuquerque, New Mexico, pp. 703–709, 1997

[8] J. Imura, K. Kobayashi and T. Yoshikawa : "Nonholonomic Control of 3 Link Planar Manipulator with a Free Joint", Pro. of the 35th IEEE Conference on Decision and Control, pp. 1435–1436, 1996

[9] H. Arai, K. Tanie and N. Shiroma : "Time-scaling Control of an Underactuated Manipulator", Proceeding of the 1998 IEEE International Conference on Robotics and Automation, Leuven, Belgium, pp. 2619–2626, 1998

[10] T. Suzuki, M. Koinuma and Y. Nakamura : "Chaos and Nonlinear Control of a Nonholonomic Free-Joint Manipulator", Proceeding of the 1996 IEEE International Conference on Robotics and Automation, Minneapolis, pp. 2668–2675, 1996

[11] A. De Luca, R. Mattone and G. Oriolo : "Stabilization of Underactuated Robots : Theory and Experiments for a Planar 2R Manipulator" ,Proceeding of the 1997 IEEE International Conference on Robotics and Automation, Albuquerque, New Mexico, pp. 3274–3280, 1997

[12] M. Fliess, J. Levine, P. Martin and P. Rouchon : " Design of trajectory stabilizing feedback for driftless flat system", Proceedings of 3rd European Control Conference, pp. 1882–1887, 1995