Grasping Optimization using a Required External Force Set

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***Abstract*— In this paper, we investigate optimal grasp points on an arbitrary shaped grasped object using a required external force set. The required external force set is given based on a task, and consists of the external forces and moments, which must be balanced by virtue of contact forces applied by a robotic hand. When the origin is in the interior of the set, a force-closure grasp is required. When the dimension of the set is one, an equilibrium grasp is required. Therefore we can investigate whatever the desired grasp is, such as when the desired grasp is a force-closure and equilibrium grasps. Also, we only have to consider the forces contained in a given required external force set, not the whole set of generable resultant forces. Furthermore, we can avoid the frame invariant problem (the criterion value changes with the change of the task (object) coordinate frame)). We consider an optimization problem from the viewpoint of decreasing the magnitudes of the contact forces needed to balance any external force and moment contained in a given required external force set. In order to solve the problem, we present an algorithm based on a branch-and-bound method. We also present some numerical examples to show the validity of our approach.**

***Index Terms*— Optimization, Grasping, Branch-and-bound method, Required external force set**

I. Introduction

Consider using a robotic hand to lift an object off a table. If the object can not be grasped with appropriate contact positions between the object and the robotic hand, the gravitational force of the object may not be able to be counteracted and the object may fail to be lifted. In the same fashion, in order to manipulate an object in a desired direction, we must grasp the object with appropriate contact positions. Selecting the contact positions is a very important issue for grasping. This article deals with optimal grasp synthesis on general objects.

Much attention has been given to investigating optimal grasp points on a grasped object [1]–[20]. This research can be classified into the following two categories: 1) research aimed at investigating optimal grasp points or regions for balancing gravitational force (namely, aimed at an equilibrium grasp) [1]–[3], 2) research aimed at investigating optimal grasp points or regions for achieving force-closure grasps [4]–[20]. Force closure implies that, if external force and moment in any direction are applied to a grasped object, the external force and moment can be resisted and the motion of the object can be restrained.

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An upper bound of the magnitude of resistible external force and moment in force-closure grasps exists. Therefore, the magnitude can be used as a criterion for grasps. Li et al. [21] evaluated a volume of the largest task ellipsoid. The ellipsoid can be embedded in a set constructed by resultant forces and moments, which can be applied to a grasped object by a robotic hand. But the computation of the volume is very complex. Using this criterion, then, makes it difficult to investigate optimal grasp points. Commencing with Mirtich et al. [4], some researchers [5]–[7] have investigated optimal grasp points, which minimize the magnitudes of contact forces needed to balance an external force. In these researches, the magnitudes of external force and moment, which must be balanced, were set to be lower than one. Since this setting was not based on a substantial reason, the authors in [4]–[7] were faced with the problem: "How do we handle the difference between the units of the external force and moment when evaluating the criterion (the magnitude of the needed contact forces)"? Markenscoff et al. [5] only evaluated force. Mirtich et al. [4] made two different criterions for force and moment and evaluated the magnitudes of contact force required to balance external force and moment separately. Mangialardi et al. [6] assumed external force and moment were applied individually to a grasped object. Wang et al. [7] indirectly evaluated the magnitudes of resistible external forces and moments by the magnitude of the contact force of the clamp. In sum, the class of the set of external forces to be balanced has been limited in these studies [4]–[7]. Also, whole resultant forces, applicable to a grasped object, must be evaluated.

In this article, we investigate optimal grasp points using a required external force set [22], [23]. This set is given based on a given task and consists of the external forces and moments, which must be balanced by virtue of contact forces applied by a robotic hand. In this case, then, the problem of criteria regarding unit does not arise(because the set is based on a task). Even when an object is manipulated, we can use this set. We can construct the set by external forces and moments which correspond to the required accelerations or the motion of the object in the manipulation. Therefore, optimal grasp points irrespective of both how to control the robotic hand and how to manipulate the object are investigated. When the origin is in the interior of the set, a force-closure grasp is required. When the dimension of the set is one, an equilibrium grasp is required. Therefore, by using the set, we can investigate any desirable grasp including force-closure and equilibrium grasps. In addition, only the forces contained in a given required external force set need be evaluated, and not the whole generable resultant force. A required external force set is based on a given task, and if the task (object) coordinate frame changes, the required external force set also changes according to the change of the task (object) coordinate frame. Criterion value does not change irrespective of the change of the task (object) coordinate frame. In other words, we can avoid the frame invariant problem (criterion value changes with the change of the task (object) coordinate frame).

In this article, we consider a problem for investigating optimal grasp points from the view point of decreasing the magnitudes of contact forces needed to balance the external forces contained in a required external force set. Commencing with Li et al. [21], there are some researches dealing with grasp synthesis from this viewpoint (task-oriented grasp synthesis). Teichmann et al. [8] considered the problem of searching the grasp points, which minimize the number of contact points, needed to balance any external force and moment contained in a given set. Zhu et al. [9], [10], [24] presented a quantitative measure of grasp, which is defined as the gauge function of a convex polyhedral set in the wrench space (this set corresponds to a required external force set). Based on the measure, they developed algorithms "for optimal grasp synthesis on polygonal objects [9]", and "for grasp synthesis on objects where contact points are located on a surface represented by one continuous function [10]". But, the used set corresponding to required external force set was assumed to contain the origin in its interior, and only a force-closure grasp was considered in the above researches [8]–[10], [24]. Therefore, the benefit of aiming at any desirable grasp was lost. In addition, Teichmann et al. did not consider friction at the contact points, and did not show any numerical examples to verify the effectiveness of their approach. Pollard [25] considered grasp synthesis in which the used set corresponding to required external force set was not assumed to contain the origin in its interior. However, she did not deal with optimization of grasp. While there have been a number of works concerning grasp synthesis, we observed that no optimal grasp synthesis on general objects yet has been done using the concept of required external force set which does not necessarily contain the origin in its interior.

To deal with this kind of optimal grasp synthesis, in this paper, we develop an algorithm based on a branch-and-bound method. In this method, a convex, polyhedral, required external force set is used. When a given required external force set is not a convex polyhedron, we use a convex polyhedron circumscribed in the convex hull of the original, given, required external force set. In this way, we can deal with any kind of required external force sets. Therefore, we can deal with any desired grasps, including force-closure and equilibrium grasps. This method can also find the global optimal solution with a small computational time, and can deal with grasp of any number of fingers.

In many cases the treatable shape of an object is restricted in the investigation of optimal grasp points. The shape was restricted to a polygon or a polyhedron in [1], [2], [5], [9], [11]–[13], [18]. Every contact point was assumed to be on a surface represented by one continuous function in [1], [2], [5], [6], [10], [11]. In this case, two problems to solve occur: one problem is to select the optimal surfaces represented by one continuous function where we will locate contact points, and the other problem is to select the optimal contact points. In this article, admissible contact points are assumed to be given by a collection of discrete candidate locations. When a set of points is not given, but the geometry of a grasped object is, we represent the geometry of the grasped object as a set of points. In this case, if every point is close enough to its adjacent point in order to attain linearity between the constraints of the problem for the neighboring points, an accurate solution can be obtained. Then, any arbitrary shaped object can be treated. Also, we can deal with the above two problems (how to select surfaces and how to select points) simultaneously.

Note that our algorithm can also be extended to the case of utilizing another criterion such as accuracy and manipulability.

This paper is organized as follows. First, we define the problem for investigating optimal grasp points using a required external force set. Next, we present an algorithm to solve the problem. Lastly, numerical examples are presented to show the effectiveness of our approach.

A. Related works

A survey about equilibrium and force-closure grasps was done by Shimoga [26]. In this survey, computation of contact forces for achieving equilibrium and force-closure grasps is discussed as well as criteria for grasping dexterity. Another survey [11] discussed the following topics: the number of fingers required for force closure, testing for force closure and planning force-closure grasps i.e., the computation of contact forces for force closure. Next, we briefly review another research aspect; namely, the investigation of grasp points for equilibrium and force-closure grasps.

As for the investigation of grasp points for equilibrium grasp, Omata [1], [2] investigated the regions of contact points on a convex polyhedral object to balance the gravitational force. Trinkle et al. [3] considered where contact points should be located, in the case of lifting an object off a table without friction using two fingers.

As for the investigation of grasp points for force closure, we can further classify the research according to whether the magnitudes of resistible external forces are a subject of interest or not. Research with no interest in the magnitudes, conducted by Nguyen [12], showed a method for investigating the regions of contact positions for achieving force closure. Chen et al. [13] presented a graphical method for investigating optimal contact positions when a 2-D and 3-D object are grasped by three and four fingers respectively, using some grasp qualities: robustness, manipulability, and task compatibility. Chen et al. [14] investigated optimal contact positions for 2-finger grasps, based on the concept of antipodal points and grasping energy function. Ponce et al. [11] showed that force closure using four fingers can be classified into concurrent grasp, pencil grasp and regulus grasp. The authors also developed a method for investigating the regions of contact positions that can construct a concurrent grasp. Van der Stappen et al. [15] presented algorithms to compute not only force-closure grasp but also second-order-immobility grasp on polygonal objects, based on a geometric search. Liu [16] presented a new sufficient and necessary condition for force-closure grasps, and based on it, developed an algorithm to compute force-closure grasps of n fingers on polygonal objects. Ding et al. [17] investigated the contact positions on a polyhedral object to achieve force closure, using a method based on a qualitative test algorithm [27] based on a ray-shooting problem. Ding et al. [18] investigated an eligible set of fixturing surfaces on a polyhedral workpiece for achieving force closure first, and then investigated the optimal fixturing positions on the eligible set, which minimize the locating error of the workpiece. Li et al. [19] investigated contact regions of fingers on polygonal objects to achieve force closure, and evaluated the stability of the contact regions by each convex polyhedron composing the contact regions. Liu et al. [20] presented an algorithm to compute 3-D force-closure grasps on objects represented by discrete points. The algorithm combines a local search process with a recursive problem decomposition strategy.

As for the research with an interest in the magnitudes of resistible external forces, Markenscoff et al. [5] investigated the grasp points which minimize the magnitudes of contact forces needed to balance gravitational force. The gravitational force was assumed to be applied to a polygonal object in the direction perpendicular to the object. The grasp was done by three fingers with frictional point contacts. Markenscoff et al. [5] also investigated the grasp points minimizing the magnitudes of contact forces needed to balance an external force. The external force was assumed to be applied to a polygonal object in any direction and its magnitude was assumed to be lower than one. The grasp was done by four fingers with frictionless point contacts. Mirtich et al. [4] first investigated the sets of grasp points, which minimize the magnitudes of contact forces needed to balance an external force, and then investigated the grasp points which minimize the magnitudes of contact forces needed to balance an external moment, among the selected sets of grasp points. The external moment was assumed to be perpendicular to the grasp plane made by the grasp points and its magnitude was assumed to be lower than one. The external force was assumed to be applied in any direction and its magnitude was assumed to be lower than one. The grasp was done by two or three fingers. Mangialardi et al. [6] investigated the grasp points which minimize the average magnitude of the normal components of contact forces required to balance external force and moment. The external force and moment were assumed to be applied to a grasped object individually in any direction with a unit magnitude. Each grasp point was assumed to be on a surface represented by one continuous function. Wang et al. [7] investigated the fixturing points which minimize the magnitudes of contact forces needed to balance external force and moment. The object was assumed to be fixed by six locators and one clamp. The locators were assumed to be able to passively apply a contact force without friction. The clamp was assumed to be able to actively apply a contact force without friction. Wang et al. considered some algorithms in the search for the optimal positions of the locators when a certain contact force is applied to the object by the clamp. Note that we can indirectly evaluate the magnitudes of resistible external forces and moments by the magnitude of the contact force of the clamp.

II. PROBLEM DEFINITION

In this section, we define a problem for investigating optimal grasp points, using a required external force set. First, we describe the target system, statics, and the frictional constraints. Second, we define the required external force set. Lastly, we formulate the problem for optimal grasp.

A. Target System

We consider the case where an arbitrary shaped rigid object is grasped by m fingers of a robotic hand. We make the following assumptions.

- Each finger makes a frictional point contact with the object at the fingertip.
- A unique normal direction at each contact point can be obtained.
- Each finger can apply any desired contact forces to the object using a force-feedback control.
- Dynamic effects are negligible. Quasi-static motion and static grasp are considered.

B. Statics and Frictional Constraints

First, we describe the relation between contact forces and external force and moment applied to the object. Let $\mathcal{C}$ be a set of all possible candidates of contact points. Let

$$\mathcal{N}=\{\boldsymbol{p}_{C_{\mathcal{N}i}}\ (i=1,2,\cdots,m)|\boldsymbol{p}_{C_{\mathcal{N}i}}\in\mathcal{C}\} \tag{1}$$

be a combination of m contact points selected from $\mathcal{C}$, where $\boldsymbol{p}_{C_{\mathcal{N}i}} \in \mathcal{R}^d$ represents the position of the ith contact point $(i=1,2,\cdots,m)$, $d=2$ in two-dimensional space, and $d=3$ in three-dimensional space.

Let $\boldsymbol{f}_{\mathcal{N}i}\in \mathcal{R}^d$ be the contact force applied to the object at the ith contact point (by the ith finger). Let $\boldsymbol{w}\in \mathcal{R}^D$ be the external force and moment applied to the object (with respect to a frame fixed at the object), where $D=3$ in two-dimensional space and $D=6$ in three-dimensional space. Then, the object is in equilibrium if the following equation holds;

$$\Sigma_{i=1}^{m}\boldsymbol{G}_{\mathcal{N}i}\boldsymbol{f}_{\mathcal{N}i}=-\boldsymbol{w}. \tag{2}$$

Here, $\boldsymbol{G}_{\mathcal{N}i}$ is

$$\boldsymbol{G}_{ij}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -(p_{C_{\mathcal{N}i_y}}-p_{o_y}) & (p_{C_{\mathcal{N}i_x}}-p_{o_x}) \end{pmatrix}\in\mathcal{R}^{D\times d}$$

in two-dimensional space, where $p_{C_{\mathcal{N}i_x}}$ and $p_{C_{\mathcal{N}i_y}}$ represent the x and y components of $\boldsymbol{p}_{C_{\mathcal{N}i}}$ respectively, and p_{o_x} and p_{o_y} represent the x and y components of $\boldsymbol{p}_o$ respectively, where $\boldsymbol{p}_o$ represents the position of the origin of the frame fixed at the object, and

$$\boldsymbol{G}_{\mathcal{N}i}=\begin{pmatrix} \boldsymbol{I} \\ [(\boldsymbol{p}_{C_{\mathcal{N}i}}-\boldsymbol{p}_o)\times] \end{pmatrix}\in\mathcal{R}^{D\times d}$$

in three-dimensional space, where $\boldsymbol{I}$ represents an identify matrix, and $[\boldsymbol{a}\times]$ represents a skew symmetric matrix equivalent to the cross product operation ($[\ \boldsymbol{a}\times\ ]\boldsymbol{b}=\boldsymbol{a}\times\boldsymbol{b}$).

Next, the frictional constraint at the ith contact point ($i = 1, 2, \cdots, m$) can be represented by

$$\mathcal{F}_{f\mathcal{N}i} = \{\boldsymbol{f}_{\mathcal{N}i} | \sqrt{t^2_{f_{\mathcal{N}i},1} + t^2_{f_{\mathcal{N}i},2}} \leq \mu_{\mathcal{N}i} n_{f_{\mathcal{N}i}}, n_{f_{\mathcal{N}i}} \geq 0\} \quad (3)$$

in three-dimensional space, where $n_{f_{\mathcal{N}i}}$ denotes the normal force component of $\boldsymbol{f}_{\mathcal{N}i}$, $t_{f_{\mathcal{N}i},1}$ and $t_{f_{\mathcal{N}i},2}$ denote the tangential force components of $\boldsymbol{f}_{\mathcal{N}i}$, and $\mu_{\mathcal{N}i}$ denotes the frictional coefficient at the ith contact point. Note that in (3), $t_{f_{\mathcal{N}i},2} = 0$ can represent the frictional constraint in two-dimensional space.

Aggregating (3) for all contact points, we obtain

$$\mathcal{F}_{f\mathcal{N}} = \{\boldsymbol{f}_{\mathcal{N}} | \boldsymbol{f}_{\mathcal{N}i} \in \mathcal{F}_{f\mathcal{N}i}, \ ^\forall i\}, \quad (4)$$

where

$$\boldsymbol{f}_{\mathcal{N}} = \left(\boldsymbol{f}^T_{\mathcal{N}1}, \boldsymbol{f}^T_{\mathcal{N}2} \cdots \boldsymbol{f}^T_{\mathcal{N}m}\right)^T.$$

C. Required External Force Set

We define the required external force set as follows.

Required External Force Set *We call an external force (moment) exerted on the object, required to be balanced for completing a given task, a "required external force". We call a set, composed of all possible required external forces, a "required external force set" $\mathcal{W}_R \subset \mathcal{R}^D$. Note that the required external force set must be given such that if we balance all the forces contained in the required external force set, the stable grasp can be guaranteed.*

D. Problem for Optimal Grasp

Based on the above discussion, we formulate a problem for investigating optimal grasp points, using a required external force set. Let $|\boldsymbol{f}_{\mathcal{N}i}| (= \sqrt{\boldsymbol{f}^T_{\mathcal{N}i}\boldsymbol{f}_{\mathcal{N}i}})$ be the norm of the contact force at the ith contact point. Let ϕ be the largest norm among $|\boldsymbol{f}_{\mathcal{N}i}|$'s ($i = 1, 2, \cdots, m$);

$$\phi = \max_i |\boldsymbol{f}_{\mathcal{N}i}|.$$

Then, ϕ has the following relation with the contact force at every contact point;

$$\phi \geq \sqrt{\boldsymbol{f}^T_{\mathcal{N}i}\boldsymbol{f}_{\mathcal{N}i}} \ (i = 1, 2, \cdots, m).$$

In general, there exist an infinite number of combinations of contact points, in which we can balance any external force and moment contained in a given, required external force set. If the external force can be balanced by small contact forces, we receive the following benefits; (1) prevention of the contact forces from deforming or destroying the object, (2) reduction of the perturbation of the resultant force and moment resulting from the perturbation of the contact forces (such perturbation can cause a disturbing force resulting from the difference between the external force and its balancing force), and (3) ability to grasp the object stably. We define the following problem.

Problem for Optimal Grasp Find the combination of contact points $\mathcal{N}^*$ which gives ρ such that

$$\rho = \min_{\mathcal{N} \in \mathcal{S}} \max_{\boldsymbol{w} \in \mathcal{W}_R} \min_{\substack{\boldsymbol{f}_{\mathcal{N}i} \\ \text{satisfy (2),(3)}}} \phi \quad (5)$$

where $\mathcal{S}$ denotes the set of all possible candidates of $\mathcal{N}$. This problem investigates the combination of contact points, which minimizes the magnitude of contact forces needed to balance any external force and moment contained in the given required external force set.

III. ALGORITHM

In this section, we present an algorithm to solve the problem (5) defined in the previous section. For the development of the algorithm, the candidate contact points are assumed to be given by a finite number of points. When not a set of points, but the geometry of a grasped object is given, we represent the geometry of the grasped object as a set of points. In this case, if every point is close enough to its adjacent point in order to attain linearity between the constraints of the problem for the neighboring points, an accurate solution can be obtained. Therefore, we have only to select the optimal contact points from the candidate contact points. Let n be the number of the candidate contact points. Then, the number of the candidate $\mathcal{N}$'s can be expressed by $C(n, m)$. (Note that $C(n, m)$ represents the number of combinations where we select m from n.) Let $\mathcal{N}_k$ be the kth $\mathcal{N}$ contained in $\mathcal{S}$ ($k = 1, 2, \cdots, C(n, m)$).

In order to solve the problem defined in (5), we use a branch-and-bound method [28], [29]. The branch-and-bound method is often applied to a problem to find an optimal solution among a finite number of feasible solutions. The feasible solutions are the solutions of subproblems into which we divide the original problem. This method finds the optimal solution by enumerating the solutions of the feasible subproblems. In the process, we eliminate the subproblems which we don't have to solve, by using a relaxed problem obtained by relaxing the constraints of the subproblem. This elimination reduces the computational time required. Note that the solution of the relaxed problem gives a lower/upper bound of the solution of the original subproblem.

First, we formulate a required external force set. Second, we define some subproblems and a relaxed problem of the problem defined in (5). Lastly, we describe the procedure of the algorithm.

A. Required External Force Set

We represent a required external force set as a convex polyhedron. However, it is possible that a given required external force set is not a convex polyhedron, as shown in Fig.1. In such a case, we will define a new required external force set that is a convex polyhedron circumscribed in the convex hull of the given original required external force set, as shown in Fig.1. The new required external force set (a convex polyhedron) contains the original, required, external force set. Therefore, if we can balance all the forces contained in the new required external force set, we can also balance all the forces contained in the original required external force set. Then, the stable grasp can be guaranteed if we balance all the forces contained in the new required external force set.

Next, let $\mathcal{A}_1$ be the set of generable resultant forces and $\mathcal{A}_2$ be a given non-convex polyhedral required external force set.

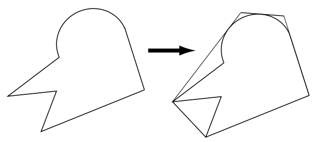


Fig. 1. How to make a convex polyhedral required external force set

We seek the grasp points whose $\mathcal{A}_1$ must contain $\mathcal{A}_2$ (the grasp points such that $\mathcal{A}_1 \supset \mathcal{A}_2$) . Denoting the convex hull of $\mathcal{A}_1$ by $\mathrm{co}\mathcal{A}_1$, we can get the relation: $\mathrm{co}\mathcal{A}_1 \supset \mathrm{co}\mathcal{A}_2$ [30]. Here, since the set of generable resultant forces is a polyhedral convex set [31], $\mathcal{A}_1 = \mathrm{co}\mathcal{A}_1$, then $\mathcal{A}_1 \supset \mathrm{co}\mathcal{A}_2$. Hence, the result for the convex hull of the non-convex polyhedral required external force set is the same as the result for a non-convex polyhedral required external force set.

$\mathrm{co}\mathcal{A}_2$ is given by:

$$\mathrm{co}\mathcal{A}_2 = \{\sum_{j=1}^{l} \lambda_j \boldsymbol{a}_j,\ \sum_{j=1}^{l} \lambda_j = 1,\ \lambda_j \geq 0,\ \boldsymbol{a}_j \in \mathcal{A}_2, l \in \mathcal{L}\}.$$

If l is finite, $\mathrm{co}\mathcal{A}_2$ becomes a convex polyhedron. When $\mathrm{co}\mathcal{A}_2$ isn't a convex polyhedron, we approximate the convex hull by a convex polyhedron circumscribed in the convex hull (let l be finite). If the approximation is accurate enough, the result for the convex polyhedral required external force set circumscribed in the convex hull of the non-convex polyhedral required external force set is almost the same as the result for a non-convex polyhedral required external force set.

Accordingly, the given required external force set $\mathcal{W}_R$ can be expressed by a convex polyhedron composed of l vertexes:

$$\begin{aligned} \mathcal{W}_R \ = \ & \{\boldsymbol{w} = \Sigma_{j=1}^{l} \lambda_j \boldsymbol{w}_{vj},\ \Sigma_{j=1}^{l} \lambda_j = 1, \\ & \lambda_j \geq 0 (j = 1, 2, \cdots, l)\} \end{aligned} \tag{6}$$

where $\boldsymbol{w}_{vj}$ denotes the jth vertex of $\mathcal{W}_R$.

B. Subproblems and the Relaxed Problem

First, we consider the case where the object is grasped at the contact points belonging to the kth (a certain) combination of contact points $\mathcal{N}_k$. Then, the following subproblem can be obtained.

Subproblem 1

$$\max_{\boldsymbol{w} \in \mathcal{W}_R} \ \min_{\boldsymbol{f}_{\mathcal{N}_k i} \ \text{satisfy (2),(3)}} \phi \tag{7}$$

If we solve Subproblem1 for every $\mathcal{N}_k$ ($k = 1, 2, \cdots, C(n, m)$), we can obtain the solution of the original problem (5).

Next, in Subproblem1 (when the object is grasped at the contact points belonging to $\mathcal{N}_k$), we consider balancing the required external force, $\boldsymbol{w}_{vj}$, which is the jth vertex of $\mathcal{W}_R$. Then, we can obtain the following subproblem of Subproblem1.

Subproblem 2

$$\begin{array}{ll} \min & \phi \\ \text{subject to} & \sqrt{\boldsymbol{f}_{\mathcal{N}_k i}^T \boldsymbol{f}_{\mathcal{N}_k i}} \leq \phi \ (i = 1, 2, \cdots, m) \\ & \Sigma_{i=1}^{m} \boldsymbol{G}_{\mathcal{N}_k i} \boldsymbol{f}_{\mathcal{N}_k i} = -\boldsymbol{w}_{vj} \\ & \boldsymbol{f}_{\mathcal{N}_k i} \in \mathcal{F}_{f\mathcal{N}_k i} \ (i = 1, 2, \cdots, m) \end{array} \tag{8}$$

Now, we consider the case where we can obtain the optimal solution of Subproblem2 for every $\boldsymbol{w}_{vj}$ with $\mathcal{N}$ fixed at $\mathcal{N}_k$ (k is fixed, $j = 1, 2, \cdots, l$). Let $\rho_{\mathcal{N}_k, vj}$ be the solution of Subproblem2 for $\boldsymbol{w}_{vj}$ where $\mathcal{N} = \mathcal{N}_k$. Then, the largest $\rho_{\mathcal{N}_k, vj}$ among $\rho_{\mathcal{N}_k, vj}$'s (k is fixed, $j = 1, 2, \cdots, l$) is the solution of Subproblem1 (for $\mathcal{N}_k$). The following is the proof.

Proof: Let $\rho_{\mathcal{N}_k}$ be the optimal value of Subproblem1 for $\mathcal{N}_k$. Let $\rho_{\mathcal{N}_k, vj\,\max}$ be the largest $\rho_{\mathcal{N}_k, vj}$ among $\rho_{\mathcal{N}_k, vj}$'s (k is fixed, $j = 1, 2, \cdots, l$). Then, it is obvious that

$$\rho_{\mathcal{N}_k} \geq \rho_{\mathcal{N}_k, vj\,\max}.$$

The outline of the proof is such that if the contact forces $\boldsymbol{f}_{\mathcal{N}_k i}$, whose magnitudes are lower than or equal to $\rho_{\mathcal{N}_k, vj\,\max}$, can satisfy (2),(3) for every $\boldsymbol{w} \in \mathcal{W}_R$, $\rho_{\mathcal{N}_k, vj\,\max}$ becomes the optimal value of Subproblem1 for $\mathcal{N}_k$.

Let $\boldsymbol{f}_{\mathcal{N}_k, j}^* = \left((\boldsymbol{f}_{\mathcal{N}_k 1, j}^*)^T \ \cdots \ (\boldsymbol{f}_{\mathcal{N}_k m, j}^*)^T\right)^T$ be the contact forces which give the $\rho_{\mathcal{N}_k, vj}$. For every contact force $\boldsymbol{f}_{\mathcal{N}_k i}$, we consider the l-sided convex polyhedron of the contact force, whose jth vertex is $\boldsymbol{f}_{\mathcal{N}_k i, j}^*$;

$$\begin{aligned} \mathcal{F}_{\mathcal{N}_k i} = \{ & \boldsymbol{f} = \Sigma_{j=1}^{l} \lambda_j \boldsymbol{f}_{\mathcal{N}_k i, j}^*, \Sigma_{j=1}^{l} \lambda_j = 1, \\ & \lambda_j \geq 0 \ (j = 1, 2, \cdots, l)\} \quad (i = 1, 2, \cdots, m). \end{aligned}$$

The relation between $\rho_{\mathcal{N}_k, vj\,\max}$ and $\boldsymbol{f} \in \mathcal{F}_{\mathcal{N}_k i}$ ($i = 1, 2, \cdots, m$) can be given by;

$$\begin{aligned} \rho_{\mathcal{N}_k, vj\,\max} & \geq \Sigma_{j=1}^{l} \lambda_j \rho_{\mathcal{N}_k, vj} \\ & \geq \Sigma_{j=1}^{l} \lambda_j \sqrt{(\boldsymbol{f}_{\mathcal{N}_k i, j}^*)^T \boldsymbol{f}_{\mathcal{N}_k i, j}^*}, \\ & \geq \sqrt{(\Sigma_{j=1}^{l} \lambda_j \boldsymbol{f}_{\mathcal{N}_k i, j}^*)^T \Sigma_{j=1}^{l} \lambda_j \boldsymbol{f}_{\mathcal{N}_k i, j}^*}, \\ & (\Sigma_{j=1}^{l} \lambda_j = 1, \lambda_j \geq 0, i = 1, 2, \cdots, m). \end{aligned}$$

In other words, the magnitude of every contact force $\boldsymbol{f} \in \mathcal{F}_{\mathcal{N}_k i}$ ($i = 1, 2, \cdots, m$) is lower than or equal to $\rho_{\mathcal{N}_k, vj\,\max}$.

Next, we consider whether $\boldsymbol{f} \in \mathcal{F}_{\mathcal{N}_k i}$ ($i = 1, 2, \cdots, m$) can satisfy (2), (3) for every $\boldsymbol{w} \in \mathcal{W}_R$.

First, since the frictional constraints (3) is convex, the contact force contained in $\mathcal{F}_{\mathcal{N}_k i}$ satisfies the frictional constraint;

$$\boldsymbol{f} \in \mathcal{F}_{f\mathcal{N} i},\ {}^\forall\ \boldsymbol{f} \in \mathcal{F}_{\mathcal{N}_k i},\ i.$$

Second, the resultant force applied by contact forces contained in $\mathcal{F}_{\mathcal{N}_k i}$ ($i = 1, 2, \cdots, m$) can be represented by

$$\begin{aligned} \mathcal{W}_r = & \{\Sigma_{i=1}^{m} \boldsymbol{G}_{\mathcal{N}_k i} \Sigma_{j=1}^{l} \lambda_j \boldsymbol{f}_{\mathcal{N}_k i, j}^*, \Sigma_{j=1}^{l} \lambda_j = 1, \lambda_j \geq 0, \\ & (j = 1, 2, \cdots, l)\}. \\ & = \{-\Sigma_{j=1}^{l} \lambda_j \boldsymbol{w}_{vj}, \Sigma_{j=1}^{l} \lambda_j = 1, \lambda_j \geq 0, (j = 1, 2, \cdots, l)\} \\ & = -\mathcal{W}_R. \end{aligned}$$

Then, $\boldsymbol{f} \in \mathcal{F}_{\mathcal{N}_k i}$ ($i = 1, 2, \cdots, m$) can satisfy (2), (3) for every $\boldsymbol{w} \in \mathcal{W}_R$. Hence, $\rho_{\mathcal{N}_k} = \rho_{\mathcal{N}_k, vj\,\max}$.

In sum, we can solve Subproblem1 by solving Subproblem2 for every $\boldsymbol{w}_{vj}$. ■

Based on a linear inequality representation of a second order cone constraint and the formulation by Buss et al. [32], the inequality constraints of Subproblem2 can be rewritten by the

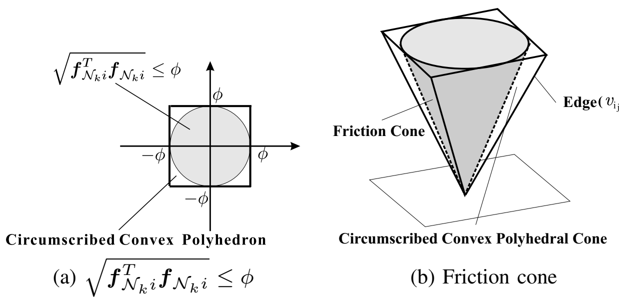


Fig. 2. Convex polyhedron and convex polyhedral cone circumscribed in the constraints of Subproblem2

following constraints with respect to the symmetric matrices $\boldsymbol{F}_{\mathcal{N}_k i}$ and $\boldsymbol{P}_{\mathcal{N}_k i}$;

$$\boldsymbol{F}_{\mathcal{N}_k i} = \begin{pmatrix} \phi \boldsymbol{I} & \boldsymbol{f}_{\mathcal{N}_k i} \\ \boldsymbol{f}^T_{\mathcal{N}_k i} & \phi \end{pmatrix} \succeq \boldsymbol{O}$$

$$\boldsymbol{P}_{\mathcal{N}_k i} = \begin{pmatrix} \mu_{\mathcal{N}_k i} n_{f_{\mathcal{N}_k i}} & 0 & t_{f_{\mathcal{N}_k i},1} \\ 0 & \mu_{\mathcal{N}_k i} n_{f_{\mathcal{N}_k i}} & t_{f_{\mathcal{N}_k i},2} \\ t_{f_{\mathcal{N}_k i},1} & t_{f_{\mathcal{N}_k i},2} & \mu_{\mathcal{N}_k i} n_{f_{\mathcal{N}_k i}} \end{pmatrix} \succeq \boldsymbol{O} \quad (9)$$

where $\boldsymbol{F}_{\mathcal{N}_k i} \succeq \boldsymbol{O}$ means $\boldsymbol{F}_{\mathcal{N}_k i}$ is a positive semidefinite matrix. Then, we can solve Subproblem2 by using a positive semidefinite program (see appendix and [33], [34]).

Next, we define a relaxed problem whose constraints are linear constraints relaxing (containing) the constraints of Subproblem2. This relaxed problem is the problem of finding $\mathcal{N}_k$'s for which we do not have to solve Subproblem1 (and Subproblem2), with a small computational time. Of course, Subproblem2 can be used to find the $\mathcal{N}_k$'s. However, since a linear programming (simplex method) requires less computational time than a positive semidefinite program, we use the relaxed problem. Note that the solution of the relaxed problem gives a lower bound of the solution for the corresponding Subproblem2; namely, that of the corresponding Subproblem1, since the constraints of the relaxed problem contain the constraints of the corresponding Subproblem2. Utilizing the lower bound, we will find $\mathcal{N}_k$'s for which we do not have to solve Subproblem1 (for details, please refer to the section III-C).

Among the constraints of Subproblem2, we approximate $\sqrt{\boldsymbol{f}^T_{\mathcal{N}_k i} \boldsymbol{f}_{\mathcal{N}_k i}} \le \phi$ by a convex polyhedron circumscribed in the constraint (Fig.2(a)). We approximate the friction cone (3) by a L-sided convex polyhedral cone circumscribed in the friction cone (Fig.2(b)) [35]. Hence, we can define the following relaxed problem.

Relaxed Problem

$$\begin{array}{ll} \min & \phi \\ \text{subject to} & \boldsymbol{e}^T_\kappa \boldsymbol{V}_{\mathcal{N}_k i} \boldsymbol{u}_{\mathcal{N}_k i} \le \phi \ (i=1,\cdots,m, \kappa=1,\cdots,d) \\ & -\boldsymbol{e}^T_\kappa \boldsymbol{V}_{\mathcal{N}_k i} \boldsymbol{u}_{\mathcal{N}_k i} \le \phi \ (i=1,\cdots,m, \kappa=1,\cdots,d) \\ & \Sigma^m_{i=1} \boldsymbol{G}_{\mathcal{N}_k i} \boldsymbol{V}_{\mathcal{N}_k i} \boldsymbol{u}_{\mathcal{N}_k i} = -\boldsymbol{w}_{vj} \end{array} \quad (10)$$

where $\boldsymbol{V}_{\mathcal{N}_k i} \in \mathcal{R}^{d \times L}$ denotes the matrix whose jth column is $\boldsymbol{v}_{\mathcal{N}_k i_j}$, which is the jth unit edge vector of the frictional convex polyhedral cone ($\boldsymbol{V}_{\mathcal{N}_k i} = (\boldsymbol{v}_{\mathcal{N}_k i_1} \cdots \boldsymbol{v}_{\mathcal{N}_k i_L})$), $\boldsymbol{u}_{\mathcal{N}_k i} (\ge \boldsymbol{o}) \in \mathcal{R}^L$ denotes the vector whose jth element represents the magnitude of the contact force in the $\boldsymbol{v}_{\mathcal{N}_k i_j}$ direction, and $\boldsymbol{e}_\kappa \in \mathcal{R}^d$ denotes the vector whose κth element is 1 and whose other elements are 0 (for example, $\boldsymbol{e}_1 = (1\ 0\ 0)^T$ in three-dimensional space). Note that the Relaxed Problem can be solved by a simplex method.

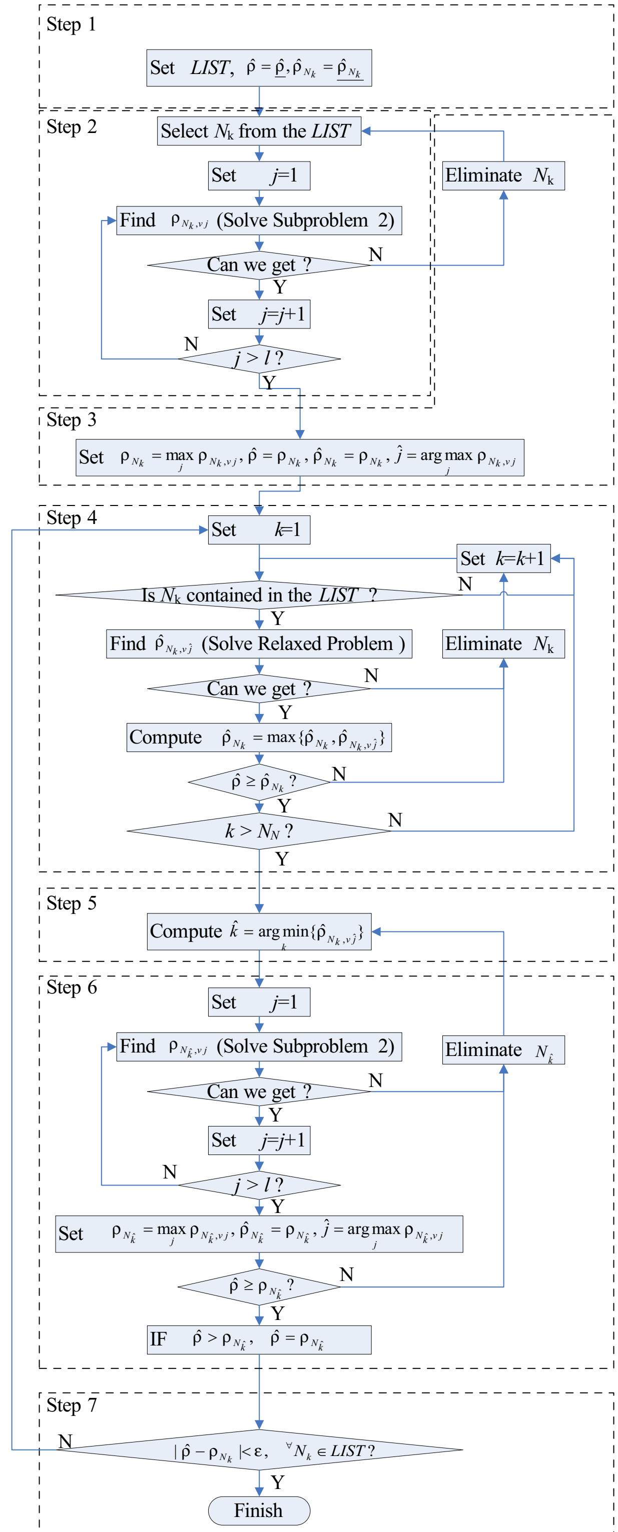


Fig. 3. Flow chart of the algorithm

C. Procedure of Algorithm

In this subsection, we describe the procedure of the algorithm to solve the problem defined in (5). Fig.3 shows the flow chart of the algorithm. We define the following nomenclatures:

$\hat{\rho}$	The (best known) feasible solution at each iteration
$\rho_{\mathcal{N}_k}$	The solution of Subproblem1 for $\mathcal{N}_k$
$\hat{\rho}_{\mathcal{N}_k}$	The (best known) lower bound of the solution of Subproblem1 at each iteration (This bound is used to find eliminable $\mathcal{N}_k$'s in Step 4.)
$\rho_{\mathcal{N}_k,vj}$	The solution of Subproblem2 for $\mathcal{N}_k$ and $\boldsymbol{w}_{vj}$
$\hat{\rho}_{\mathcal{N}_k,vj}$	The solution of the Relaxed Problem for $\mathcal{N}_k$ and $\boldsymbol{w}_{vj}$
$LIST$	The list of feasible $\mathcal{N}_k$'s
N_N	The number of the feasible $\mathcal{N}_k$'s ($N_N \leq C(n,m)$)
$\underline{\rho}$	The lower bound value of $\hat{\rho}$ (the initial value of $\hat{\rho}$)
$\underline{\hat{\rho}_{\mathcal{N}_k}}$	The lower bound value of $\hat{\rho}_{\mathcal{N}_k}$ (the initial value of $\hat{\rho}_{\mathcal{N}_k}$)

Step 1 Put all feasible (candidate) $\mathcal{N}_k$'s into the $LIST$. Set $\hat{\rho}= \underline{\rho}$ and $\hat{\rho}_{\mathcal{N}_k}= \underline{\rho_{\mathcal{N}_k}}$ $(k = 1, 2, \cdots, N_N)$.

Step 2 First, we search the first eligible value of $\hat{\rho}$, solving Subproblem1 at a certain $\mathcal{N}_k$. Select a certain $\mathcal{N}_k$ from the $LIST$ and solve Subproblem1 for the $\mathcal{N}_k$ as shown in Fig.4(a). Here is how to solve Subproblem1. First, compute the solutions of the all Subproblem2s which are the subproblems of Subproblem1 for the $\mathcal{N}_k$ (Compute $\rho_{\mathcal{N}_k,vj}$'s (k is fixed, $j = 1, 2, \cdots, l$)). If there exists at least one Subproblem2 which has no solution, the Subproblem1 has no solution. If there exist the solutions of the all Subproblem2s, the largest $\rho_{\mathcal{N}_k,vj}$ among the solutions $\rho_{\mathcal{N}_k,vj}$'s (k is fixed, $j = 1, 2, \cdots, l$) becomes the solution of the Subproblem1 ($\rho_{\mathcal{N}_k} = \max_j\{\rho_{\mathcal{N}_k,vj}\}$).

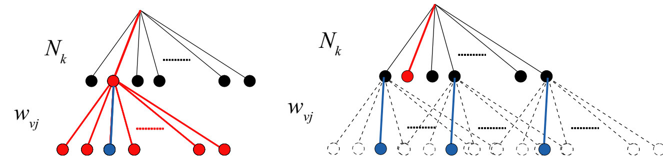
Step 3 If we can get the solution of the Subproblem1 in Step 2 ($\rho_{\mathcal{N}_k} = \max_j\{\rho_{\mathcal{N}_k,vj}\}$), substitute the solution into $\hat{\rho}$ ($\hat{\rho}= \rho_{\mathcal{N}_k}$), also substitute the solution into $\hat{\rho}_{\mathcal{N}_k}$ ($\hat{\rho}_{\mathcal{N}_k} = \rho_{\mathcal{N}_k}$). Let $\hat{j}$ be the number of the largest $\rho_{\mathcal{N}_k,vj}$($\hat{j}$=arg max$_j$ $\{\rho_{\mathcal{N}_k,vj}\}$). Let $\boldsymbol{w}_{v\hat{j}}$ be the $\boldsymbol{w}_{vj}$ which gives the solution $\rho_{\mathcal{N}_k}$. If we can not get the solution, we eliminate the $\mathcal{N}_k$ from the $LIST$ and go back to Step2. Note that we select $\boldsymbol{w}_{v\hat{j}}$ for the search, supposing the corresponding $\rho_{\mathcal{N}_k,v\hat{j}}$ at every $\mathcal{N}_k$ will be most possibly the largest among $\rho_{\mathcal{N}_k,vj}$'s ($j = 1, 2, \cdots, l$).

Step 4 Solve the Relaxed Problem where $\boldsymbol{w}_{vj} = \boldsymbol{w}_{v\hat{j}}$, and at every $\mathcal{N}_k$ contained in the $LIST$ (j is fixed to be $\hat{j}$. k is the number of each $\mathcal{N}_k$ contained in the $LIST$) as shown in Fig.4(b). In other words, we fix $\boldsymbol{w}_{vj}$ and search the eliminable $\mathcal{N}_k$'s. If we cannot get the solution at a certain $\mathcal{N}_k$, we must eliminate the $\mathcal{N}_k$ from the $LIST$. If we can get the solution $\hat{\rho}_{\mathcal{N}_k,v\hat{j}}$ at a certain $\mathcal{N}_k$, compute $\hat{\rho}_{\mathcal{N}_k} = \max\{\hat{\rho}_{\mathcal{N}_k},\ \hat{\rho}_{\mathcal{N}_k,v\hat{j}}\}$. Therefore, we compute the (best known) lower bound of the solution of Subproblem1 for the $\mathcal{N}_k$ at this moment. If $\hat{\rho} < \hat{\rho}_{\mathcal{N}_k}$, eliminate the $\mathcal{N}_k$ from the $LIST$. At the end of this subsection we present the proof for why the eliminations are approved.

Step 5 Let $\mathcal{N}_{\hat{k}}$ be the $\mathcal{N}_k$ at which $\hat{\rho}_{\mathcal{N}_k,v\hat{j}}$ is the least among $\hat{\rho}_{\mathcal{N}_k,v\hat{j}}$'s at all $\mathcal{N}_k$'s contained in the $LIST$ ($\hat{k}$=arg min$_k$ $\{\hat{\rho}_{\mathcal{N}_k,v\hat{j}}\}$). Note that we select $\mathcal{N}_{\hat{k}}$ for the search, supposing $\mathcal{N}_{\hat{k}}$ will most possibly be the optimal $\mathcal{N}_k$.

Step 6 Solve Subproblem1 for the $\mathcal{N}_{\hat{k}}$ as shown in Fig.4(a). If we can get the solution$\rho_{\mathcal{N}_{\hat{k}}}$ ($\rho_{\mathcal{N}_{\hat{k}}} = \max_j\{\rho_{\mathcal{N}_{\hat{k}},vj}\}$), substitute the solution into $\hat{\rho}_{\mathcal{N}_k}$. Also, substitute the $\boldsymbol{w}_{vj}$, which gives the solution, into $\boldsymbol{w}_{v\hat{j}}$. Note that $\hat{j}$=arg max$_j$ $\{\rho_{\mathcal{N}_{\hat{k}},vj}\}$. If we cannot get the solution or $\hat{\rho} < \rho_{\mathcal{N}_{\hat{k}}}$, eliminate the $\mathcal{N}_{\hat{k}}$ from the $LIST$ and go back to Step5. If $\hat{\rho} > \rho_{\mathcal{N}_{\hat{k}}}$, $\hat{\rho} = \rho_{\mathcal{N}_{\hat{k}}}$.

Step 7 If we can get the relation $|\hat{\rho} - \rho_{\mathcal{N}_k}| < \epsilon$ (ϵ denotes an arbitrary small positive value) at all $\mathcal{N}_k$'s contained in the $LIST$, finish the loop. Otherwise, go back to Step4.



(a) Step 2(6) (b) Step 4

Fig. 4. The diagram of Step 2(6) and Step 4

Remark: This algorithm belongs to a branch-and-bound algorithm. Since the algorithm should conceptually enumerate all possible solutions of the finite number of problems, it always convergences at the global optimum. In Step 4, we evaluate the alternatives of the solutions and judge whether the $\mathcal{N}_k$ is non-optimal or not, within a small computational time. Next, we eliminate $\mathcal{N}_k$'s which are revealed to be non-optimal. This fast elimination in Step 4 enables (users) to reduce the search domain, and to avoid an exhaustive search, thus shortening the computational time.

The following proof illustrates why the eliminations are approved in Step 4.

Proof: The set of the constraints of Relaxed Problem contains that of Subproblem2. Therefore, if there is no solution to the relaxed problem, then there is also no solution to the corresponding Subproblem2. If there is a solution to the relaxed problem, then the solution to the relaxed problem gives a lower bound of the solution of the corresponding Subproblem2: $\hat{\rho}_{\mathcal{N}_k,vj} \leq \rho_{\mathcal{N}_k,vj}$. Let $\mathcal{G}_{j,\mathcal{N}_k}$ be the set of j's whose corresponding $\rho_{\mathcal{N}_k,vj}$ has already been solved at the iteration. Then, the following relation holds:

$$\begin{aligned}\hat{\rho}_{\mathcal{N}_k} &= \max_{j\in\mathcal{G}_{j,\mathcal{N}_k}} \{\hat{\rho}_{\mathcal{N}_k,vj}\} \leq \max_j\{\hat{\rho}_{\mathcal{N}_k,vj}\} \\ &\leq \max_j\{\rho_{\mathcal{N}_k,vj}\} = \rho_{\mathcal{N}_k}.\end{aligned}$$

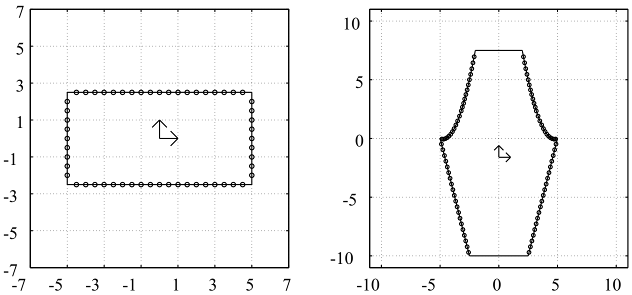
If $\hat{\rho} < \hat{\rho}_{\mathcal{N}_k}$, then $\hat{\rho} < \rho_{\mathcal{N}_k}$. Therefore, we can see that the $\mathcal{N}_k$ is not optimal, and we can eliminate the $\mathcal{N}_k$. ■

IV. Numerical Examples

In order to show the effectiveness of our approach, we show some numerical examples in this section.

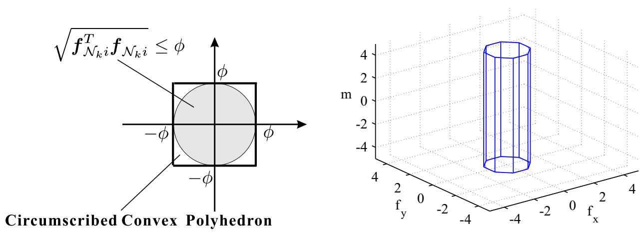
A. Examples in Two-Dimensional Space

We show the target objects and the candidate contact points in Fig.5. In this figure, the points on the objects indicate the candidate contact points. The number of the candidate contact points is 56 in Case I and 86 in Case II. Note that the candidate



(a) Case I (b) Case II

Fig. 5. Target objects and candidate contact points



(a) Case I (b) Case II

Fig. 6. Required external force set

contact points are sampled such that the candidate contact points are uniformly distributed. The object coordinate frame is located at the geometric center of each object (the reference frame is also located at the same geometric center).

Letting f_x and f_y be the x and y components of the external force, respectively, and m be the external moment, we set the required external force set as follows:

$$\mathcal{W}_R = \{ \begin{pmatrix} f_x \\ f_y \\ m \end{pmatrix} = \sum_{j=1}^{8} (\lambda_{1j} \begin{pmatrix} w_{xj} \\ w_{yj} \\ 0.8\gamma sg \end{pmatrix} + \lambda_{2j} \begin{pmatrix} w_{xj} \\ w_{yj} \\ -0.8\gamma sg \end{pmatrix}), \sum_{i=1}^{2}\sum_{j=1}^{8} \lambda_{ij} = 1,\ \lambda_{ij} \geq 0\} \quad (11)$$

$$w_{xj} = \frac{sg}{\cos(\frac{\pi}{8})} \cos(\frac{\pi}{8} + \frac{\pi}{4} j)$$

$$w_{yj} = \frac{sg}{\cos(\frac{\pi}{8})} \sin(\frac{\pi}{8} + \frac{\pi}{4} j)$$

where s denotes the area of each object, g=0.01 denotes the specific gravity of each object, and γ denotes the distance between the geometric center of each object and the vertex closer to the geometric center than any other vertexes. Here, we consider the following case;

- The gravitational force (external force) can be applied to the object in any arbitrary direction, resulting from the motion of the robotic arm equipped with the robotic hand.
- The external moment can be applied to the object resulting from the displacement of the position of the center of gravity. The position of the center of gravity can move within the circle. The center of the circle is the geometric center, and the radius of the circle is 0.8γ.

TABLE I

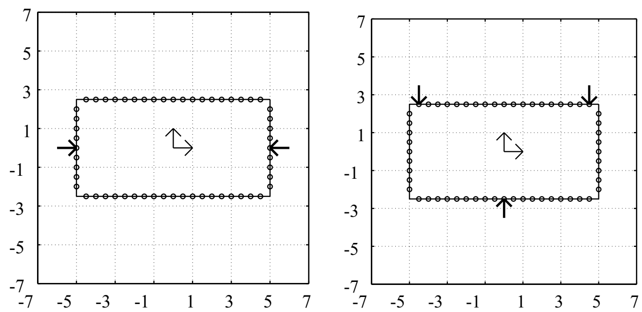
ρ IN THE PROBLEM DEFINED IN (5)

(a) Case I

μ	0.1	0.3	0.5
2 fingers	3.71	1.38	0.919
3 fingers	3.13	0.978	0.724

(b) Case II

μ	0.1	0.3	0.5
2 fingers	35.1	6.44	3.92
3 fingers	6.21	2.77	2.01



(a) 2 fingers (b) 3 fingers

Fig. 7. Optimal grasp points in Case I

In this case, a convex hull of the $\mathcal{W}_R$ can be written by

$$\mathcal{W}_R = \{(f_x,\ f_y,\ m)^T | \sqrt{f_x^2 + f_y^2} \leq sg, \quad -0.8\gamma sg \leq m \leq 0.8\gamma sg\}.$$

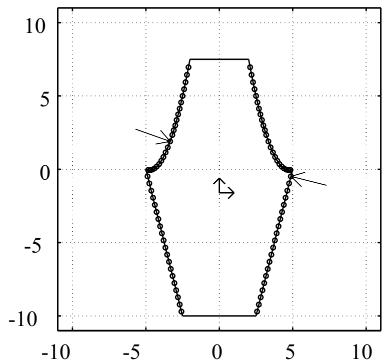
And, we approximated $\sqrt{f_x^2 + f_y^2} \leq sg$ by a regular octahedron circumscribed in the sets (see Fig.6). Thus, we can obtain the $\mathcal{W}_R$ given by (11).

The computation was done in the cases where the number of fingers is 2 and 3 and where the frictional coefficient is 0.1, 0.3, and 0.5. When we obtain some optimal $\mathcal{N}_k$'s by disregarding symmetrically arranged $\mathcal{N}_k$'s, we searched the (optimal) $\mathcal{N}_k$ which minimizes the criterion

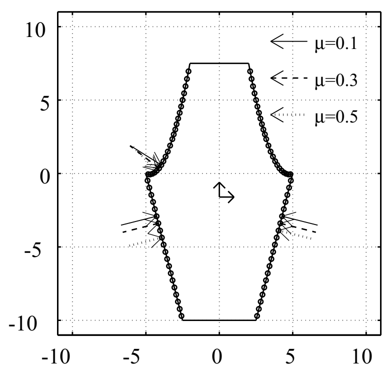
$$\max_{\boldsymbol{w} \in \mathcal{W}_R} \min_{\boldsymbol{f}_{\mathcal{N}i} \text{ satisfy (2),(3)}} \sqrt{\sum_{i=1}^{m} \boldsymbol{f}_{\mathcal{N}i}^T \boldsymbol{f}_{\mathcal{N}i}}$$

among the obtained optimal $\mathcal{N}_k$'s. By this search, we can obtain the grasp points which minimize the norm of contact forces needed to balance any external force and moment contained in the required external force set.

The results of the optimal combination of contact points are shown in Fig.7 and Fig.8. The results regarding the optimal value of the problem defined in (5), ρ, are shown in Table I. In Fig.7 and Fig.8, (a) shows the results in the case where the object is grasped by 2 fingers, (b) shows the results in the case where the object is grasped by 3 fingers, and the arrows show the obtained optimal grasp points. Note that we show either of the obtained 2 optimal combinations of contact points in the case where their arrangements are symmetric. Note also that we show only one optimal combination of contact points



(a) 2 fingers



(b) 3 fingers

Fig. 8. Optimal grasp points in Case II

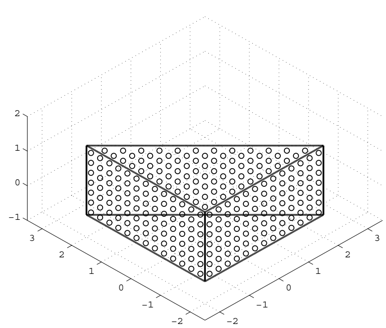


Fig. 9. Target object and candidate contact points in three-dimensional space ($n = 429$)

in Fig.7(a), (b), and Fig.8(a), since the optimal combination of contact points was the same for every magnitude of the frictional coefficient.

From Fig.8(b), we can see that the optimal grasp points approach the symmetric arrangement with the increase of the magnitude of the frictional coefficient. Here, the symmetric arrangement means any angles, between any two of three lines connecting "the geometric center of the object" and "the optimal grasp points", are $120°$. From Table I, we can see that the magnitude of contact forces needed to balance the required external force, becomes small with increases of the magnitude of the frictional coefficient and the number of fingers.

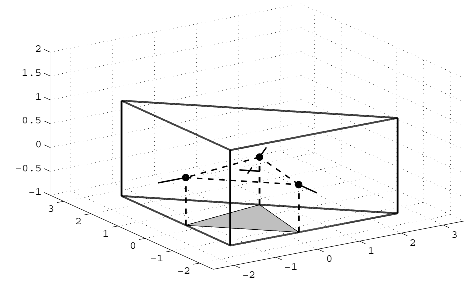


Fig. 10. Optimal grasp points in three-dimensional space ($n = 429$)

B. Example in Three-Dimensional Space

The target object and the candidate contact points are shown in Fig.9. The target object is a triangular prism whose base is a right isosceles triangle ($4 \times 4 \times 4\sqrt{2}$). The points on the object indicate the candidate contact points. Note that Fig.9, for ease of viewing, does not show the candidate contact points on the bottom face and on the back side (this side corresponds to the hypotenuse of the top/bottom face). We included the candidate contact points on the bottom face in the same way as those on the top face. We included the candidate contact points on the back side in the same way as those on the front sides. The number of the candidate points is 429 (78 for the top and bottom faces, 91 for the every front side and the back side). Note that the candidate contact points are sampled such that the candidate contact points are uniformly distributed. The object coordinate frame is located at the geometric center of the object (the reference frame is also located at the same geometric center). We used a 16-sided frictional convex polyhedral cone in the Relaxed Problem. We set the following required external force set, considering the similar situation of the case in two-dimensional space.

$$\begin{aligned}\mathcal{W}_R = \{(\boldsymbol{f}^T,\ \boldsymbol{m}^T)^T | -sg \leq f_i \leq sg,\\ -0.8\gamma sg \leq m_i \leq 0.8\gamma sg\ (i = 1, 2, 3)\}\end{aligned} \quad (12)$$

where $\boldsymbol{f}$ and $\boldsymbol{m}$ denote the external force and moment respectively, f_i and m_i denote the ith components of $\boldsymbol{f}$ and $\boldsymbol{m}$ respectively, s denotes the volume of the object, g=0.01 denotes the specific gravity of the object, and γ denotes the distance between the geometric center of the object and the face which is closer to the geometric center than any other faces. Here, we consider the following case;

- The gravitational force (external force) can be applied to the object in any arbitrary direction, resulting from the motion of the robotic arm equipped with the robotic hand.
- The external moment can be applied to the object resulting from the displacement of the position of the center of gravity. The position of the center of gravity can move within the sphere. The center of the sphere is the geometric center, and the radius of the sphere is 0.8γ.

In this case, a convex hull of $\mathcal{W}_R$ can be written by

$$\mathcal{W}_R = \{(\boldsymbol{f}^T,\ \boldsymbol{m}^T)^T | \sqrt{\boldsymbol{f}^T\boldsymbol{f}} \leq sg, \sqrt{\boldsymbol{m}^T\boldsymbol{m}} \leq 0.8\gamma sg\}.$$

And, we approximated $\sqrt{\boldsymbol{f}^T\boldsymbol{f}} \leq sg$ and $\sqrt{\boldsymbol{m}^T\boldsymbol{m}} \leq 0.8\gamma sg$ by cubes circumscribed in the sets. Thus, we can obtain the

$\mathcal{W}_R$ given by (12).

The number of the fingers was set to 3. The frictional coefficient was set to 0.3.

The result of the optimal combination of contact points is shown in Fig.10. In Fig.10, the arrows, whose tip is a black sphere, show the obtained optimal grasp points. X shows the position of the origin of the reference frame. The gray triangle shows the projection drawing of the triangle, made by the three optimal grasp points, on the bottom face of the object. The positions of the obtained optimal grasp points were $(0.3590,-4/3,0)^T$, $(2/3,2/3,0)^T$, and $(-4/3, 0.3590, 0)^T$. The optimal value of the problem (5), ρ, was 0.688.

From this result, we can see that the optimal point on the back side is located at the projection of the X point (origin) on the back side, and the optimal point on the every front side is located at the position shifting a little toward the back side from the projection of the X point (origin) on the front side. For the confirmation, we compute ρ in the case where we grasp the object at the projection of the X point (origin) on every side. The ρ was 0.718, which is bigger than 0.688 (the optimal value).

C. *Efficiency of the algorithm*

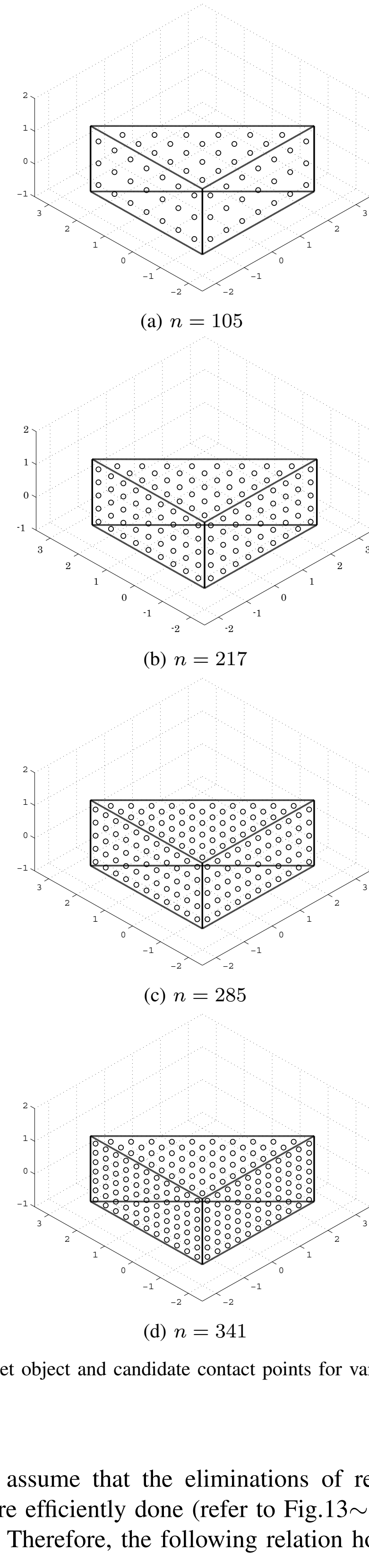
Branch-and-bound-algorithm completely does not guarantee a smaller computational time than an exhaustive enumerating search. However, in a practical sense, the eliminations of redundant enumeration through child problems dramatically reduce the computational time. Here, we discuss the efficiency of our algorithm, from the view point of iteration number, elimination number of $\mathcal{N}_k$'s, sampling effect of contact points, and reduction of computational time.

First, we consider the iteration number of our algorithm, using the following nomenclatures:

- N_{sdp_j} The number of $\mathcal{N}_k$'s where the number of the calculated Subproblem2s is $j(\leq l)$.
- N_{sim_j} The number of $\mathcal{N}_k$'s where the number of the calculated Relaxed Problems is $j(\leq l)$.
- n_{sdp} The number of constraints of Subproblem2 ($= 3m + m(d+1)$).
- n_{sim} The number of constraints of the Relaxed Problem ($= 2md + D$).

Note that if j of $N_{sdp_j}(N_{sim_j})$ is smaller than l, the corresponding $\mathcal{N}_k$'s are eliminated before all possible corresponding Subplblem2s (relaxed Problems) are calculated. When we do the exhaustive enumerating search (enumerating all possible solutions of Subproblem1(2)), N_{sdp_l} is equal to N_N.

Since the number of iterations of a positive semidefinite program is given by $O(9(n_{sdp})^{2.5})$ where setting $\boldsymbol{X}_0 = \boldsymbol{S}_0 = 10\boldsymbol{I}$ and $\varepsilon = 1 \times 10^{-7}$ (see appendix), the total iteration number for calculating Subproblem2s in our algorithm is given by $O(\sum_{j=1}^{l} N_{sdp_j} j 9(n_{sdp})^{2.5})$. Similarly, since the number of iterations of a linear program (simplex method) is given by $O(2n_{sim})$, the total iteration number needed to calculate the Relaxed Problems in our algorithm is given by $O(\sum_{j=1}^{l} N_{sim_j} j 2n_{sim})$. The total iteration number for the exhaustive enumerating search is given by $O(N_N l 9(n_{sdp})^{2.5})$.



(a) $n = 105$

(b) $n = 217$

(c) $n = 285$

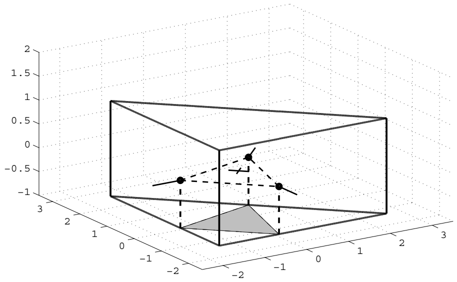
(d) $n = 341$

Fig. 11. Target object and candidate contact points for various n's

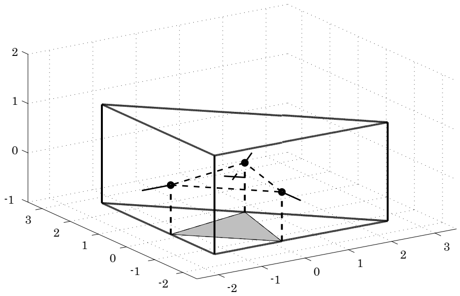
Here we assume that the eliminations of redundant enumerations are efficiently done (refer to Fig.13∼16 at the later discussion). Therefore, the following relation holds:

$$\sum_{j=1}^{l} N_{sdp_j} j \ll N_{sdp_l} l = N_N l.$$

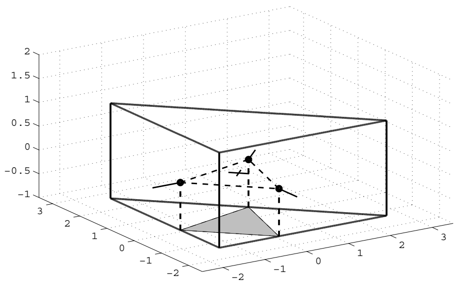
In this case, the relation between the total iteration numbers "for Subproblem2s in our algorithm" and "for an exhaustive



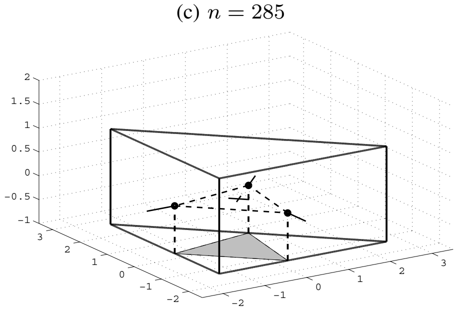
(a) $n = 105$



(b) $n = 217$



(c) $n = 285$



(d) $n = 341$

Fig. 12. Optimal grasp points for various n's

enumerating search" is given by:

$$O(\sum_{j=1}^{l} N_{sdp_j} j 9(n_{sdp})^{2.5}) \ll O(N_N l 9(n_{sdp})^{2.5}). \qquad (13)$$

On the other hand, since $\sum_{j=1}^{l} N_{sim_j} j < N_N l$, $2n_{sim} \ll 9(n_{sdp})^{2.5}$, the relation between the total iteration numbers "for Relaxed Problems in our algorithm" and "for an exhaustive enumerating search" is given by:

$$O(\sum_{j=1}^{l} N_{sim_j} j 2n_{sim}) \ll O(N_N l 9(n_{sdp})^{2.5}). \qquad (14)$$

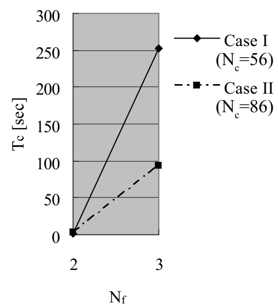
The total iteration number for our algorithm is given by the summation of the iteration numbers for Subproblem2s and Relaxed Problems. From (13) and (14), since both the iteration numbers are much smaller than that for an exhaustive enumerating search, the total iteration number for our algorithm is much smaller than that for an exhaustive enumerating search. In addition, the computational time at each iteration for a Relaxed Problem is much smaller than that for a Subproblem2. Hence, total computational time for our algorithm is much smaller than that for an exhaustive enumerating search.

Next, we show the efficiency of our algorithm using an elimination number of $\mathcal{N}_k$'s, the number of candidate contact points (related with sampling effect of contact points), and computational time in the practical numerical examples as described in sections IV-A and IV-B. Our algorithm was implemented in C/C++ and the calculations were done on a Dell Dimension 8400 computer (CPU: Pentium 4 3.4GHz, Memory: 1GB).

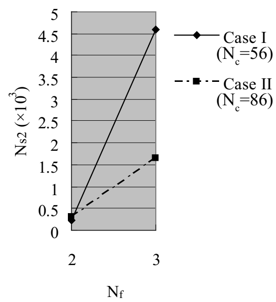
Fig.13 (a), (b) and (c), respectively, show the computational time, the number of calculated Subproblem2s, and the number of calculated Relaxed Problems in each case of two-dimensional examples (the fictional coefficient is 0.3). From Fig.13 (c) (and (b)), we can see that the number of calculated Relaxed Problems is much larger than the number of calculated Subproblem2s, and then that the eliminations of redundant enumeration are efficiently done in every case. But, from Fig.13 (a), we can see that the computational time does not always depend on the number of candidate contact points. It is also clear that the computational time increases with increase of the number of fingers.

To investigate the sampling effect of contact points, we calculated for various numbers of candidate contact points in the case of the three-dimensional example in section IV-B. Fig.11 (a) $\sim$ (d) and Fig.9 show the target object and the candidate contact points for each case. Fig.12 (a) $\sim$ (d) and Fig.10 show the obtained optimal contact points for each case. Fig.14 (a), (b) and (c), respectively, show the computational time, the number of calculated Subproblem2s, and the number of calculated Relaxed Problems for each case. Simultaneously, we computed Subproblem2 in place of the Relaxed Problem to find $\mathcal{N}_k$'s which we do not have to solve for Subproblem1. Fig.15 (a) and (b), respectively, show the computational time and the number of calculated Subproblem2s for each case in this example.

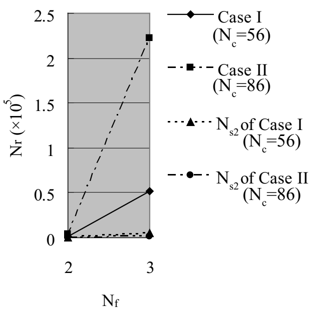
From Fig.12 (a) $\sim$ (d) and Fig.10, we can see that the optimal contact points are almost the same and therefore, the sampling effect of contact points is small with respect to optimal contact points. From Fig.14, we can see that the computational time increases with increase of the number of candidate contact points, and that the number of calculated Relaxed Problems is much larger than the number of calculated Subproblem2s. This result shows that the eliminations of redundant enumeration work effectively in our algorithm. Comparing Fig.14 with Fig.15, it is clear that the computational time is dramatically reduced for the eliminations of redundant enumeration using the Relaxed Problem (see Fig.15 (a). In Fig.15 (a), the reference value represents the results when using our original algorithm (Fig.14 (a)), and the obtained value represents the results when using eliminations by Subproblem2.). It means that the eliminations of redundant



(a) T_c[sec]

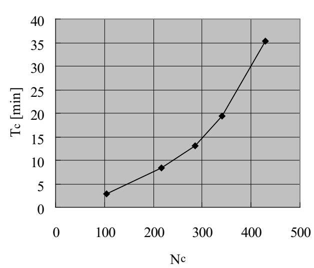


(b) N_{s2}

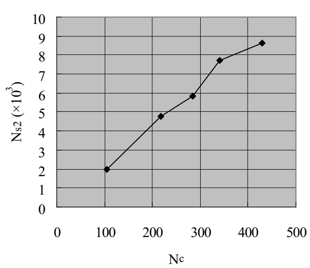


(c) N_r

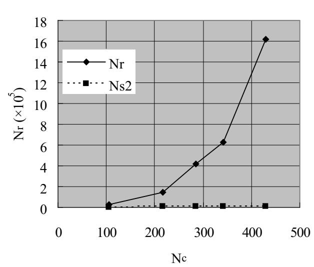
Fig. 13. Performances for the cases in two-dimensional space; (a) T_c: Computational time [sec], (b) N_{s2}: Number of calculated Subproblem2s, and (c) N_r: Number of calculated Relaxed Problems. Note that N_c denotes the number of candidate contact points, and N_f denotes the number of fingers.



(a) T_c[min]



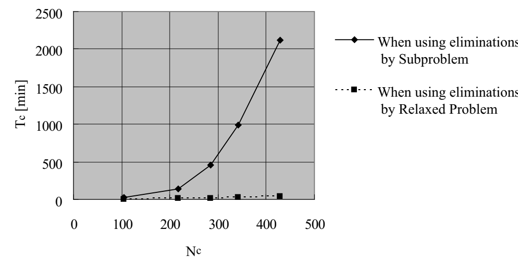
(b) N_{s2}



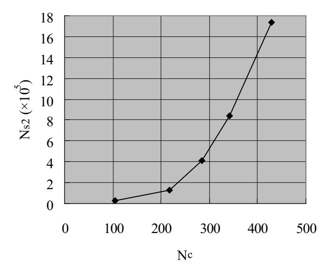
(c) N_r

Fig. 14. Performances for various numbers of candidate contact points (three-dimensional space); (a) T_c: Computational time [min], (b) N_{s2}: Number of calculated Subproblem2s, and (c) N_r: Number of calculated Relaxed Problems. Note that N_c denotes the number of candidate contact points.

enumeration by the Relaxed Problem work effectively in our algorithm.

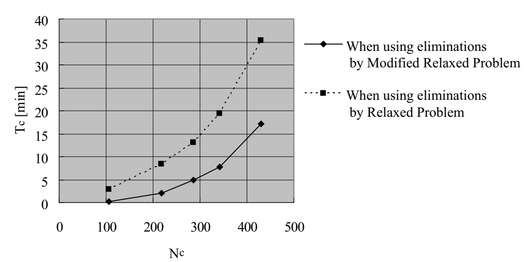


(a) T_c[min]

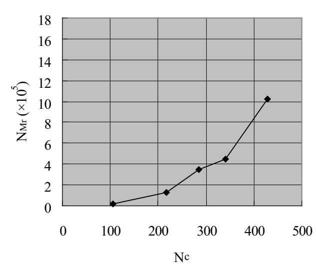


(b) N_{s2}

Fig. 15. Performances for the cases using eliminations by Subproblem2 (three-dimensional space); (a) T_c: Computational time [min], and (b) N_{s2}: Number of calculated Subproblem2s. Note that N_c denotes the number of candidate contact points.



(a) T_c[min]



(b) N_{Mr}

Fig. 16. Performances for the cases where the Modified Relaxed Problem is used in place of Subproblem2 and Relaxed Problem (three-dimensional space); (a) T_c: Computational time [min], and (b) N_{Mr}: Number of calculated Modified Relaxed Problems. Note that N_c denotes the number of candidate contact points.

In our approach, the number of the candidate combinations of contact points is $C(n,m)$, and the number becomes large with increase of the candidate contact points or the number of fingers. Here, we introduce a way to further reduce the computational time. We consider the following Modified Relaxed

Problem:

Modified Relaxed Problem

$$\begin{array}{ll}\min & \phi \\ \text{subject to} & \boldsymbol{e}_\kappa^T \boldsymbol{V}'_{\mathcal{N}_k i} \boldsymbol{u}'_{\mathcal{N}_k i} \leq \phi/\sqrt{2}\ (i=1,\cdots,m,\kappa=1,\cdots,d) \\ & -\boldsymbol{e}_\kappa^T \boldsymbol{V}'_{\mathcal{N}_k i} \boldsymbol{u}'_{\mathcal{N}_k i} \leq \phi/\sqrt{2}\ (i=1,\cdots,m,\kappa=1,\cdots,d) \\ & \Sigma_{i=1}^m \boldsymbol{G}_{\mathcal{N}_k i} \boldsymbol{V}'_{\mathcal{N}_k i} \boldsymbol{u}'_{\mathcal{N}_k i} = -\boldsymbol{w}_{vj}\end{array}$$

where $\boldsymbol{V}'_{\mathcal{N}_k i} \in \mathcal{R}^{d\times L}$ denotes the matrix whose jth column is $\boldsymbol{v}'_{\mathcal{N}_k i_j}$ which is the jth unit edge vector of the L-sided convex polyhedral cone inscribed in the friction cone ($\boldsymbol{V}'_{\mathcal{N}_k i}=(\boldsymbol{v}'_{\mathcal{N}_k i_1} \cdots \boldsymbol{v}'_{\mathcal{N}_k i_L})$), and $\boldsymbol{u}'_{\mathcal{N}_k i}(\geq \boldsymbol{o}) \in \mathcal{R}^L$ denotes the vector whose jth element represents the magnitude of contact force in the $\boldsymbol{v}_{\mathcal{N}_k i_j}$ direction.

The Modified Relaxed Problem is obtained by linearizing the constraints of Subproblem2. Note that the constraints of the Modified Relaxed Problem is contained in the constraints of Subproblem2, and thus, the solution of the Modified Relaxed Problem is a feasible solution of Subproblem2, different from the solution of the Relaxed Problem (note that the solution of the Relaxed Problem is possibly not a feasible solution of Subproblem2).

By using the Modified Relaxed Problem in place of both Subproblem2 and the Relaxed Problem in our algorithm, we can not only reduce the computational time, but also get a sub-optimal feasible solution. In this case, instead of seeking the solution to Subproblem1 for every $\mathcal{N}_k$, we seek the solution to the following problem, which linearizes original Subproblem1:

Linearized Subproblem1

$$\max_{\boldsymbol{w}_{vj}\in\mathcal{W}_R} \min_{\boldsymbol{f}_{\mathcal{N}_k i}\in\Omega'} \phi,$$

$$\Omega'=\left\{\begin{array}{c} \boldsymbol{e}_\kappa^T \boldsymbol{V}'_{\mathcal{N}_k i} \boldsymbol{u}'_{\mathcal{N}_k i} \leq \phi/\sqrt{2}\ (i=1,\cdots,m,\kappa=1,\cdots,d) \\ -\boldsymbol{e}_\kappa^T \boldsymbol{V}'_{\mathcal{N}_k i} \boldsymbol{u}'_{\mathcal{N}_k i} \leq \phi/\sqrt{2}\ (i=1,\cdots,m,\kappa=1,\cdots,d) \\ \Sigma_{i=1}^m \boldsymbol{G}_{\mathcal{N}_k i} \boldsymbol{V}'_{\mathcal{N}_k i} \boldsymbol{u}'_{\mathcal{N}_k i} = -\boldsymbol{w}_{vj}\end{array}\right\}.$$

Fig.16 (a) and (b), respectivelly, show the computational time and the number of calculated Modified Relaxed Problems for each case. The obtained optimal contact points for each case are same as those as shown in Fig.12 (a) $\sim$ (d) and Fig.10. Hence, we can get enough optimal grasp points by this Modified Relaxed Problem- based algorithm. Comparing Fig.14 (a) with Fig.16 (a), it is clear that the Modified Relaxed Problem- based algorithm requires a smaller computational time (see Fig.16 (a). In Fig.16 (a), the reference value represents the results when using our original algorithm (Fig.14 (a)), and the obtained value represents the results when using Modified Relaxed Problem- based algorithm.). Thus, if only (sub-)optimal contact points are needed and an optimal value is not needed, the (sub-)optimal contact points can be obtained in smaller computational time using the Modified Relaxed Problem- based algorithm.

The other way to further reduce the computational time is by first reducing the candidate contact points and searching the optimal grasp points, and second, by searching the optimal grasp points among the original candidate contact points near the obtained optimal grasp points. By following this way to reduce computational time, we get approximate optimal grasp points in a smaller computational time. The efficiency of this method is clear from Fig.13$\sim$16.

V. Conclusion

In this article, we have investigated optimal grasp points on a grasped object using the concept of a required external force set. By using the required external force set, we can deal with any desired grasps, including force-closure and equilibrium grasps. Also, we only have to consider the forces contained in a given required external force set, and not the whole generable resultant forces. In addition, we can avoid the frame invariant problem (criterion value changes with the change of the task (object) coordinate frame). We have considered an optimization problem from the viewpoint of decreasing the magnitudes of the contact forces needed to balance any required external force contained in a given required external force set. In order to solve the problem, we have presented an algorithm based on a branch-and-bound method. We have also presented some numerical examples in order to show the validity of our approach. In addition, we discuss the efficiency of our algorithm, from the view point of iteration number, elimination number of combinations of contact points, sampling effect of contact points and reduction of computational time.

In this article, we have assumed that each finger makes a frictional point contact with the object. However, even when each finger makes a soft-finger contact with the object, we can still use our approach due to an expression of the frictional constraints corresponding to (9) as proposed by Buss et al. [32].

In addition, even when we would like to use another criterion such as manipulability and accuracy, we can still use our approach if we can find the value of the criterion for a certain required external force at certain grasp points by utilizing a programming method such as a sequential quadratic programming.

Appendix I
Positive Semidefinite Program [33]

Here we summarize a positive semidefinite program. A positive semidefinite program is formulated as follows:

$$\begin{array}{ll}\min & \sum_{i=1}^m b_i y_i \\ \text{subject to} & \sum_{i=1}^m \boldsymbol{A}_i y_i - \boldsymbol{C} = \boldsymbol{S}, \boldsymbol{S} \succeq \boldsymbol{O}\end{array} \quad (15)$$

where $\boldsymbol{A}_i$, $\boldsymbol{C}$, $\boldsymbol{S}$ are $n\times n$ symmetric matrices. The dual problem of this problem is given by

$$\begin{array}{ll}\max & \boldsymbol{C}\bullet\boldsymbol{X} \\ \text{subject to} & \boldsymbol{A}_i\bullet\boldsymbol{X}\ (i=1,\cdots,m),\ \boldsymbol{X}\succeq\boldsymbol{O}\end{array} \quad (16)$$

where $\boldsymbol{X}$ is a $n\times n$ symmetric matrix, $\boldsymbol{X}\bullet\boldsymbol{Y}$ denotes the inner product of $\boldsymbol{X}$ and $\boldsymbol{Y}$, namely, the trace of $\boldsymbol{X}^T\boldsymbol{Y}$, $\mathrm{Tr}\boldsymbol{X}^T\boldsymbol{Y}$. These problems can be regarded as an extension of a standard linear program. Similarly as a linear program, the problem is solved by a primal-dual interior-point method in a polynomial time. The number of iteration is given by $O(n^{2.5}\log(\boldsymbol{X}_0\bullet\boldsymbol{Y}_0\varepsilon^{-1}))$ where $\boldsymbol{X}_0$ and $\boldsymbol{Y}_0$ are the initial value of $\boldsymbol{X}$ and $\boldsymbol{Y}$, respectively, and ε is the accuracy of an optimal solution.

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